COLLEGE ALGEBRA

HARDING AND MULLINS

COLLEGE ALGEBRA

A SERIES OF MATHEMATICAL TEXTS

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PREFACE

The present volume contains the material that is usually included in a semester's course in college algebra. No radical departure is made from the material that is more or less standard for such a course. It is recognized, however, that the needs of colleges differ and for this reason an effort has been made to present the topics in such a way as to give the text the greatest flexibility. Thus for the most part many of the chapters are independent, so that the omission of portions of chapters, or of certain entire chapters, will not break the continuity of the course.

One special feature of the text is the frequent use of the graphical method for the purpose of clarifying the processes of algebra, rather than on its own account.

Another distinctive feature of this book is the introduction of the derivative. This treatment of the derivative is direct and simple, and the graphical illustration will make it easily understood. The sole purpose of introducing this bit of the calculus is to enable the student to find with ease the maximum and minimum values of simple polynomial functions and to pave the way for Newton's graphical method for approximating the roots of equations of higher degrees.

Still another important feature is the abundance of review material. Many colleges and universities now admit students who have had only one year of high school algebra. In view of this fact, the first part of the text contains a complete review of elementary algebra. This makes it possible for the instructor in freshman mathematics to spend a few weeks reviewing high school algebra before he takes up those subjects that belong to college algebra. The first part of the book, however, is not a mere collection of material intended

to be used as a review, but is complete and is arranged in logical order.

Rather extensive tables, taken chiefly from THE MACMILLAN TABLES, are given, including short trigonometric tables, extensive interest tables, and a complete table of squares, cubes, square roots, and cube roots. It is hoped that these tables will be of interest and value.

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DECEMBER, 1927.

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COLLEGE ALGEBRA

CHAPTER I

INTRODUCTION

1. The Number-System of Algebra.* As a natural result of counting the number of objects in a group, *positive integers*, or *whole numbers*, were first introduced.

These numbers were sufficient for expressing the result of adding, or multiplying two numbers, and of subtracting a smaller from a larger number, but division was possible only under certain conditions. For example, a group of 12 objects could be divided equally among two, three, four, six, or twelve people. There was, however, no integer which would express the result of dividing a group of 12 objects into five equal groups. This led to the introduction of *fractional numbers* such as 12/5, 7/2, 2/3, $\cdots$, which are expressed as the quotient of two integers. The number system then contained integers, fractions, and the number zero.

It is evident that any integer may be written in fractional form. The same is true of mixed numbers, and also of both terminating and repeating decimal fractions.

For example,

$$3 = \tfrac{3}{1} = \tfrac{6}{2} = \tfrac{9}{3}, \text{ etc.}, \qquad 2\tfrac{1}{3} = \tfrac{7}{3},$$
$$4.125 = 4\tfrac{1}{8} = \tfrac{33}{8}, \text{ and} \qquad 0.2855\cdots = \tfrac{2}{7}.$$

A number which may be expressed as the quotient of two integers is called a ***rational number.*** Most of the numbers which are used in arithmetic are of this type.

* In this introductory chapter no attempt will be made to consider all of the topics treated in elementary algebra. The student who needs a more thorough review of high-school algebra should consult any current text on that subject.

There are certain other numbers, most of which were introduced as a result of the study of geometry, which cannot be expressed in this way. For example, if the side of a square is one foot long, the number which expresses the length of the diagonal in feet is represented by the symbol $\sqrt{2}$; and the number of square feet in the area of a circle whose radius is one foot in length, is represented by the Greek letter π. All such numbers are known as ***irrational numbers.***

Although an irrational number cannot be expressed exactly as the quotient of two integers, it is possible to find a rational number which will approximate its value with any desired degree of accuracy.

Thus

$\sqrt{2} = 1.4,$ or $1.41,$ or $1.414,$ or $1.4142,$ or $1.41421,$ and

$\pi = 3.1,$ or $3.14,$ or $3.142,$ or $3.1416,$ or 3.14159

Rational and irrational numbers together constitute the number-system of arithmetic.

In algebra the number-system described above is used, but the results are made more general by the substitution of various general symbols (usually the letters of the alphabet x, y, z, a, b, c, etc.) for the definite symbols of arithmetic. In view of the fact that we are now dealing with letters instead of definite numbers, we will frequently meet such an expression as $x - y$, where we do not know which of the two numbers is the greater. In order that the operation of subtraction may always be possible in such cases we introduce a new type of number known as a ***negative number.*** Thus, we say that the result of subtracting 7 from 5 is -2.

The number-system of algebra, therefore, contains both rational numbers (positive and negative) and irrational numbers (positive and negative). With the general symbols representing these numbers, the four fundamental operations

are always possible and the results can always be expressed by combinations of these general symbols, except that division by zero must be excluded. A further expansion of this number-system will be found in § 16.

2. **The Four Fundamental Operations.** The application of these operations to algebraic expressions is illustrated by the following examples.

EXAMPLE 1. Find the sum of $5x^2 + ax + 3$, $2x^2 - bx + c$, and $7x^2 + 2x + 4$.

$$\begin{array}{l} 5x^2 + ax + 3 \\ 2x^2 - bx + c \\ 7x^2 + 2x + 4 \\ \hline 14x^2 + (a - b + 2)x + (7 + c) \end{array}$$

EXAMPLE 2. Subtract $ax^2 + 2xy - 5y^2$ from $bx^2 - 5xy + 4y^2$.

$$\begin{array}{l} bx^2 - 5xy + 4y^2 \\ ax^2 + 2xy - 5y^2 \\ \hline (b - a)x^2 - 7xy + 9y^2 \end{array}$$

EXAMPLE 3. Multiply $2x^3 - 4x^2 + 5x - 7$ by $x^2 + 2x - 3$.

$$\begin{array}{l} 2x^3 - 4x^2 + 5x - 7 \\ x^2 + 2x - 3 \\ \hline 2x^5 - 4x^4 + 5x^3 - 7x^2 \\ \quad + 4x^4 - 8x^3 + 10x^2 - 14x \\ \quad\quad - 6x^3 + 12x^2 - 15x + 21 \\ \hline 2x^5 \quad - 9x^3 + 15x^2 - 29x + 21 \end{array}$$

EXAMPLE 4. Divide $12x^4 - 17x^3 + 16x^2 - 7x + 2$ by $3x^2 - 2x + 1$.

$$\begin{array}{l|l} 12x^4 - 17x^3 + 16x^2 - 7x + 2 & 3x^2 - 2x + 1 \\ 12x^4 - 8x^3 + 4x^2 & \hline 4x^2 - 3x + 2 \\ \hline - 9x^3 + 12x^2 - 7x & \\ - 9x^3 + 6x^2 - 3x & \\ \hline 6x^2 - 4x + 2 & \\ 6x^2 - 4x + 2 & \\ \hline \end{array}$$

Addition and subtraction of algebraic expressions may be performed in the manner indicated below. A parenthesis preceded by a minus sign may be removed provided the sign of every term in the parenthesis is changed. If the parenthesis is preceded by a plus sign, no change of sign is required. Thus

$$\begin{aligned}(5x^2 + ax + 3) + (2x^2 - bx + c) + (7x^2 + 2x + 4)\\ &= 5x^2 + ax + 3 + 2x^2 - bx + c + 7x^2 + 2x + 4\\ &= (5 + 2 + 7)x^2 + (a - b + 2)x + (3 + c + 4)\\ &= 14x^2 + (a - b + 2)x + (7 + c),\end{aligned}$$

and

$$\begin{aligned}(bx^2 - 5xy + 4y^2) - (ax^2 + 2xy - 5y^2)\\ &= bx^2 - 5xy + 4y^2 - ax^2 - 2xy + 5y^2\\ &= (b - a)x^2 + (-5 - 2)xy + (4 + 5)y^2\\ &= (b - a)x^2 - 7xy + 9y^2.\end{aligned}$$

If two expressions are arranged according to the same order of powers of some common letter, the processes of multiplication and division may be abridged by writing only the numerical coefficients of the terms. This is known as the *method of detached coefficients.*

Example 5. Multiply $x^2 + 3x - 2$ by $2x^2 - x + 4$.

$$\begin{array}{r} 1 + 3 - 2 \\ 2 - 1 + 4 \\ \hline 2 + 6 - 4 \\ \quad - 1 - 3 + 2 \\ \qquad\quad 4 + 12 - 8 \\ \hline 2 + 5 - 3 + 14 - 8 \end{array}$$

The exponent of x in the first term is obviously 4. Hence the product is $2x^4 + 5x^3 - 3x^2 + 14x - 8$.

EXAMPLE 6. Divide $x^4 - x^3 - 6x^2 + 15x - 9$ by $x^2 + 2x - 3$.

$$\begin{array}{r|l} 1 - 1 - 6 + 15 - 9 & 1 + 2 - 3 \\ 1 + 2 - 3 & \hline 1 - 3 + 3 \\ \hline \quad - 3 - 3 + 15 & \\ \quad - 3 - 6 + 9 & \\ \hline \qquad 3 + 6 - 9 & \\ \qquad 3 + 6 - 9 & \\ \hline \end{array}$$

Hence the quotient is $x^2 - 3x + 3$.

When using the method of detached coefficients care must be taken to supply all missing terms with zero coefficients.

EXERCISES

Find the sum of:

1. $4x - 20z$, $8x + 9y$, $5z - 3y$.
2. $10x^3 + 6x + 7$, $3x^3 - 5x^2 - 6$, $2x^3 - 2x + 3$.
3. $5m^3 - 3m^2 + 2m$, $m^3 - m^2$, $m^2 - m + 1$, $m^3 - 2m^2 - m - 2$.
4. $8a^3 - a^2b + 6ab^2 + 3b^3$, $-9a^3 + 4a^2b - 7ab^2 - 6b^3$, $a^3 + a^2b - ab^2 + b^3$, $a^3 - a^2b - ab^2 + 3b^3$.
5. $12(a + b) - 15(b + c)$, $6(b + c) - 17(c + a)$, $-5(c + a) - 3(a + b)$.
6. $16(a + b - c) - 3(x - y) + 2(a - b)$, $2(x - y) - 3(a - b) + (a + b - c)$, $7(a - b) + 4(x - y) - 3(a + b - c)$.

Subtract the first expression from the second:

7. $3ax^3 - 7a^2x^2 + 3a^3x - 2a^4$, $x^4 - ax^3 - a^2x^2 - a^3x - 2a^4$.
8. $7xy^2 - y^3 - 3x^2y + 5x^3$, $8x^3 + 7x^2y - 3xy^2 - y^3$.
9. $(c + a)x + (a + b)y + (b + c)z$, $(a + b)x + (b + c)y + (c + a)z$.
10. Subtract the sum of $4a - 9b + 6c$ and $2a + 8b - 11c$ from $7a - 3b - 5c$.
11. From the sum of $a - 2b + 4m - 2n$ and $3a + 3b + 6m - 5n$ subtract the sum of $-2a + 5b - 8m - 3n$ and $6a - 4b + 18m - 4n$.

Simplify each of the following expressions:

12. $x + (2x - y) - (4x - 7y) - (6y - x)$.

13. $3a - 3b - 5c - (-4b + 3c - 4a) - (4a + b - 8c)$.

14. $-(m - 3n + 4p) + (-3m + 4n - 3p) - (-6m - 7n)$.

Collect in parentheses the coefficients of x, y, and z:

15. $ax + by + cz + 3x - 4z + 5y - bx - cy$.

16. $4x - 3z + 5y - 6x + 2y - 7x + 6z - 4y$.

17. $-2x - 3y + 2z - 4y + 5x - 4z + 7y + 2z - 2x$.

18. What expression must be added to $3a - 4b + 6c$ to give $5a - 10b + 3c$?

19. What expression must be subtracted from $x^2 + 2xy + 3y^2$ to give $x^2 - 3xy - 5y^2$?

If $a = x + y + z$, $b = x + y - z$, $c = x - y + z$, and $d = y + z - x$, find the value of

20. $a + b + c + d$.

21. $a + b + c - d$.

22. $a + b - c + d$.

23. $a - b + c + d$.

Multiply the following expressions, using detached coefficients when possible:

24. $x + y$, $2x - 3y$.

25. $2a - 9$, $2a + 8$.

26. $ab - cd$, $ab + 2cd$.

27. $x^2 - x + 1$, $2x^2 - x$.

28. $a^2 + 2ab - b^2$, $2a^2 - ab + b^2$.

29. $x^4 + 2x^3 + x^2 - 4x - 11$, $x^2 - 2x + 3$.

30. $a^2 - ab + b^2 + a + b + 1$, $a + b - 1$.

Divide the first expression by the second, using detached coefficients when possible:

31. $2x^2 + x - 6$, $2x - 3$.

32. $2a^2 - 3a + 1$, $a - 1$.

33. $x^3 - 1$, $x^2 + x + 1$.

34. $a^3 + 8$, $a + 2$.

35. $x^6 - 6x^4 + 9x^2 - 4$, $x^2 - 1$.

36. $a^3 + b^3 - c^3 + 3abc$, $a + b - c$.

37. $x^6 - 6x^4y^2 + 9x^2y^4 - 4y^6$, $x^3 - 2x^2y - xy^2 + 2y^3$.

3. **Type Products and Quotients.** The student will save much labor by becoming familiar with certain types of products and quotients which occur very frequently in algebraic problems.

$$(x + y)(x - y) = x^2 - y^2,$$
$$(x + y)^2 = x^2 + 2xy + y^2,$$
$$(x + a)(x + b) = x^2 + (a + b)x + ab,$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$$
$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz,$$
$$\frac{x^2 - y^2}{x + y} = x - y, \qquad \frac{x^2 - y^2}{x - y} = x + y,$$
$$\frac{x^3 - y^3}{x - y} = x^2 + xy + y^2, \qquad \frac{x^3 + y^3}{x + y} = x^2 - xy + y^2.$$

These quotients are special cases of the following general formulas. If n is any positive integer, then

$$\frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1},$$
$$\frac{x^n + y^n}{x + y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots + (-1)^{n-1}y^{n-1},$$

n odd,

$$\frac{x^n - y^n}{x + y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots + (-1)^{n-1}y^{n-1},$$

n even.

EXERCISES

Write by inspection the value of the following:

1. $(p + 4q)(p - 4q)$.
2. $(2 + rs)(2 - rs)$.
3. $(3x - 10y)(3x + 10y)$.
4. $(2a + 3b)^2$.
5. $(x + 4)(x - 1)$.
6. $(mn + 3)(mn - 5)$.
7. $(3x - 8)(3x + 7)$.
8. $(a^2 - 5b^2)^2$.
9. $(2a^2 + ab - 3b^2)^2$.
10. $(m + n + p)^3$.

Simplify the following:

11. $(2 - x)(2 + x)(4 + x^2)(16 + x^4)$.

12. $(x^2 - y^2 - z^2)(x^2 + y^2 + z^2)$.

13. $(m + n + p)^3 - (m + n - p)^3$.

14. $(a + b + c)^2 + (a - b - c)^2 + (a - b + c)^2 + (a + b - c)^2$.

15. $(a + b)^3 - 3(a + b)^2(a - b) + 3(a + b)(a - b)^2 - (a - b)^3$.

Write the following quotients:

16. $\dfrac{8x^3 - 1}{2x - 1}$.

17. $\dfrac{27a^3 - b^3}{3a - b}$.

18. $\dfrac{36x^4 - 49}{6x^2 + 7}$.

19. $\dfrac{4 - (a + b)^2}{2 - a - b}$.

20. $\dfrac{1 - 16x^4}{1 - 2x}$.

21. $\dfrac{a^{12} + 27}{a^4 + 3}$.

22. $\dfrac{x^7 + 1}{x + 1}$.

23. $\dfrac{x^{3m} + 8}{x^m + 2}$.

24. $\dfrac{a^{6m} - 27}{a^{2m} - 3}$.

4. **Synthetic Division.** The operation of dividing a polynomial by a binomial of the type $x - a$ may be very much simplified by making use of a contracted process known as ***synthetic division.***

Let us divide $x^3 + 3x^2 - 13x + 15$ by $x - 2$, using detached coefficients.

$$
\begin{array}{rl|l}
1 + 3 - 13 + 15 & & 1 - 2 \\
\cancel{1} - 2 & & \hline
\underline{\quad\quad} & & 1 + 5 - 3 \text{ Quotient} \\
\quad 5 & & \\
\quad \cancel{5} - 10 & & \\
\underline{\quad\quad\quad} & & \\
\quad\quad - 3 & & \\
\quad\quad -\cancel{5} + 6 & & \\
\underline{\quad\quad\quad\quad} & & \\
\quad\quad\quad\quad 9 & \text{Remainder} &
\end{array}
$$

The quotient is $x^2 + 5x - 3$ and the remainder is 9.

If we wish to shorten this process as much as possible, we will, of course, omit the first term of each partial product,

since it is the same as the number immediately above. The coefficient of the first term of the divisor is 1 in every example of the type we are considering and consequently may be omitted. We then have

$$\begin{array}{rrrrr|l} 1 & +3 & -13 & +15 & -2 & \\ -2 & & & & \hline & 1+5-3 \text{ Quotient} \\ \hline & 5 & & & & \\ & -10 & & & & \\ \hline & -3 & & & & \\ & & +6 & & & \\ \hline & & & 9 & & \text{Remainder} \end{array}$$

We now notice that the first terms of the successive dividends reading down the diagonal are coefficients in the quotient. Hence we need not retain the quotient. Raising the remaining numbers into a horizontal line, we now have

$$\begin{array}{rrrrr|l} 1 & +3 & -13 & +15 & -2 & \\ & -2 & -10 & +6 & & \\ \hline 1 & +5 & -3 & +9 & & \end{array}$$

The first three numbers in the last line are the coefficients in order of the quotient and the last number is the remainder. The numbers in this line will be unchanged if we change the sign of each partial product in the second line and add it to, instead of subtracting it from, the corresponding term of the dividend. This may be accomplished by changing the second term of the divisor $x - 2$ from -2 to $+2$. Thus we have

$$\begin{array}{rrrrr|l} 1 & +3 & -13 & +15 & 2 & \\ & 2 & +10 & -6 & & \\ \hline 1 & +5 & -3 & +9 & & \end{array}$$

The contracted process of dividing any polynomial in x by $x - a$, which we have called synthetic division, may now be stated as follows:

Write the coefficients in order according to descending powers of x.

The first term of the quotient will be the same as the first term of the dividend.

Multiplying this number by a and adding the product to the second term of the dividend gives the second term of the quotient.

Multiplying this number by a and adding the product to the third term of the dividend gives the third term of the quotient, etc.

This process is to be continued until the last term of the dividend is reached.

The final sum is the *remainder*.

EXAMPLE 1. Divide $2x^4 - 5x^3 - 4x - 5$ by $x - 3$.

Since the term in x^2 is missing we must supply it with a coefficient 0.

$$\begin{array}{r} 2 - 5 + 0 - 4 - 5 \,\big|\, \underline{3} \\ \quad 6 + 3 + 9 + 15 \\ \hline 2 + 1 + 3 + 5 + 10 \end{array}$$

Hence the quotient is $2x^3 + x^2 + 3x + 5$ and the remainder is 10.

EXAMPLE 2. Divide $x^5 + 1$ by $x + 1$.

$$\begin{array}{r} 1 + 0 + 0 + 0 + 0 + 1 \,\big|\, \underline{-1} \\ \quad -1 + 1 - 1 + 1 - 1 \\ \hline 1 - 1 + 1 - 1 + 1 + 0 \end{array}$$

Hence the quotient is $x^4 - x^3 + x^2 - x + 1$ and there is no remainder. That is, $x^5 + 1$ is exactly divisible by $x + 1$. Compare with the formulas of § 3.

EXERCISES

Divide the first number by the second, using synthetic division:

1. $x^2 - 7x + 12,\ x - 3.$
2. $3x^2 + 8x + 5,\ x + 1.$
3. $4a^2 + 13a - 12,\ a + 4.$
4. $a^3 - 7a + 6,\ a - 1.$
5. $2x^3 + 4x^2 + 2x + 13,\ x + 2.$
6. $x^2 + 3ax - 28a^2,\ x - 4a.$
7. $a^4 - 1,\ a + 1.$
8. $a^6 - 1,\ a - 1.$
9. $x^4 - 6x^3 + 2x^2 + 10x + 4,\ x - 2.$
10. $2x^5 + 5x^4 - 3x^3 + 2x^2 + 5x + 2,\ x + 3.$

11. Show how to apply the process of synthetic division in finding the quotient of $2x^3 + x^2 - 14x + 12$ by $2x - 3$.

5. **Factoring.** To find the factors of any polynomial is to find two or more expressions whose product is the given polynomial. Thus factoring is the reverse of multiplication and the type products of § 3 when written in reverse order become formulas for factoring. We shall set down several of these types for reference.

(1) *Monomial factors.* If each term of an expression contains a common monomial factor, this monomial is a factor of the expression.

Thus

$$ax + 2ay + 3a = a(x + 2y + 3).$$

(2) *The form* $x^2 - y^2$. The difference of the squares of two numbers is always equal to the product of the sum and the difference of the two numbers.

Thus

$$16a^2 - 25b^2 = (4a)^2 - (5b)^2 = (4a + 5b)(4a - 5b),$$

and

$$\begin{aligned} a^4 - 3a^2b^2 + b^4 &= a^4 - 2a^2b^2 + b^4 - a^2b^2 = (a^2 - b^2)^2 - (ab)^2 \\ &= (a^2 - b^2 + ab)(a^2 - b^2 - ab) \\ &= (a^2 + ab - b^2)(a^2 - ab - b^2). \end{aligned}$$

(3) *The forms* $x^2 \pm 2xy + y^2$. This is evidently the square of the binomial $x \pm y$.

Thus

$$4m^2 + 12mn + 9n^2 = (2m)^2 + 2 \times 2m \times 3n + (3n)^2 = (2m + 3n)^2.$$

(4) *The form* $x^2 + mx + n$. Suppose the two factors are denoted by $x + a$ and $x + b$. Then

$$x^2 + mx + n = (x + a)(x + b) = x^2 + (a + b)x + ab.$$

The problem now is to find two numbers a and b whose sum is m and whose product is n. Such numbers always exist, though they may be imaginary. They may be found either by inspection or by solving a quadratic equation (§ 17).

Thus if we wish to factor $x^2 - 2x - 35$, we must find two numbers whose sum is -2 and whose product is -35. These numbers are -7 and $+5$. Hence

$$x^2 - 2x - 35 = (x - 7)(x + 5).$$

Also

$$(x^2 - 4x)^2 - 2(x^2 - 4x) - 15 = (x^2 - 4x - 5)(x^2 - 4x + 3) = (x - 5)(x + 1)(x - 3)(x - 1).$$

It is evident that the form (3) is a special case of form (4).

(5) *The form* $ax^2 + bx + c$. An expression of this type may sometimes be factored by inspection. Suppose the factors are $px + q$ and $rx + s$. Then

$$ax^2 + bx + c = (px + q)(rx + s) = prx^2 + (ps + qr)x + qs.$$

We must therefore divide a into two factors p, r, and c into two factors q, s, such that $ps + qr = b$. If these numbers cannot be found by inspection, the expression may be factored by solving a quadratic equation (§ 17).

Consider the expression $12x^2 + 7x - 10$. The factors of 12 are 12, 1 and 6, 2 and 4, 3. Those of 10 are 10, 1 and 5, 2. We now try different combinations of these factors connected by plus and minus signs, until we strike the right combination. Thus

$$\begin{aligned}(12x - 1)(x + 10) &= 12x^2 + 119x - 10,\\ (6x + 2)(2x - 5) &= 12x^2 - 26x - 10,\\ (3x + 5)(4x - 2) &= 12x^2 + 14x - 10,\\ (3x + 2)(4x - 5) &= 12x^2 - 7x - 10,\\ (3x - 2)(4x + 5) &= 12x^2 + 7x - 10.\end{aligned}$$

Hence the factors of the given expression are $3x - 2$ and $4x + 5$.

(6) *The forms* $x^3 \pm y^3$. A glance at the type quotients of § 3 shows that

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2),$$

and

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2).$$

Thus

$$\begin{aligned}27m^3 - 8n^3 &= (3m)^3 - (2n)^3\\ &= (3m - 2n)(9m^2 + 6mn + 4n^2),\end{aligned}$$

and

$$a^6 + b^6 = (a^2)^3 + (b^2)^3 = (a^2 + b^2)(a^4 - a^2b^2 + b^4).$$

(7) *The forms* $x^n \pm y^n$. By making use of the results of § 3 it may be easily seen that, if n is a positive integer,

$x^n - y^n$ has a factor $x - y$, always,
$x^n - y^n$ has a factor $x + y$, when n is even,
$x^n + y^n$ has a factor $x + y$, when n is odd.
$x^n + y^n$ has no linear factor when n is even.

Thus

$$\begin{aligned}x^5 - y^5 &= (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4),\\ x^4 - y^4 &= (x + y)(x^3 - x^2y + xy^2 - y^3),\\ x^5 + y^5 &= (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4).\end{aligned}$$

It is sometimes necessary to make use of two or more of the foregoing formulas before an expression is completely factored. For example,

$$a^6 - b^6 = (a^3 + b^3)(a^3 - b^3), \quad \text{by (2)},$$
$$= (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2), \quad \text{by (6)}.$$

EXERCISES

Factor each of the following expressions.

1. $mn - m^2 - n^2 + mn$.
2. $a^5 + ab^4 - a^4b - b^5$.
3. $a^2x + a^2y + a^2z + x + y + z$.
4. $t^3 + at^2 + ct^2 + bt + act + ab$.
5. $a^2 - 5a - 84$.
6. $x^2 + xy - 132y^2$.
7. $x^2 + 14xy - 15y^2$.
8. $x^2y^2 - 2xy - 35$.
9. $8x^2 - 22x - 21$.
10. $9a^2b^4 - 30ab^2c + 25c^2$.
11. $p^2 - 10pq + 25q^2$.
12. $10 - x - 3x^2$.
13. $a^2b^2 - abc - 110c^2$.
14. $x^2 - 20xy + 91y^2$.
15. $20a^2 - ab - 99b^2$.
16. $15a^2 + 44ab - 20b^2$.
17. $31x^2 + 23xy - 8y^2$.
18. $2xy^2 - 50x$.
19. $20a^2 + 9ab - 18b^2$.
20. $5 + 9y - 18y^2$.
21. $6x^2 - x - 2$.
22. $36a^2 - 132ab + 121b^2$.
23. $4m^2n^2 + 36mn + 81$.
24. $225x^6 + 30x^3 + 1$.
25. $10x^3y - 5x^2y^2 - 5xy^3$.
26. $c^3 + 27d^3$.
27. $x^6 + 6x^3 - 27$.
28. $x^{12} - 4$.
29. $729x^6 - 1$.
30. $a^6 - b^6$.
31. $a^8 - 256$.
32. $a^8 - b^8$.
33. $a^8b^8c^8 - x^8$.
34. $a^{10} - b^{10}$.
35. $(a + b)^5 - 32$.
36. $x^3 + 3x^2y + 3xy^2 + y^3 - z^3$.
37. $a^3 - b^3 + c^3 + 3b^2c - 3bc^2$.
38. $27(x + y)^3 - 1331a^3b^3$.
39. $(x + 1)^3 - (y + 1)^3$.
40. $(5a^2 - 4b)^2 - (3a^2 - 2b)^2$.
41. $a^{2n} - b^{2n}$.
42. $4a^{2m} - 9b^{2n}$.

43. $a^2 + b^2 - x^2 - y^2 - 2ab + 2xy$.

44. $(3a - b)^2 - 2(3a - b)(a - 2b) + (a - 2b)^2$.

45. $(2m - 3n)^2 - 5(2m - 3n)(a + 3b) - 14(a + 3b)^2$.

46. $x^3 + 8y^3 - z^3 - 8p^3 + 6x^2y - 6z^2p + 12xy^2 - 12zp^2$.

47. $a^2 - 4cx - 2ax + 2ac + 2bx - 3ad + 6dx - ab$.

48. $81x^2 - m^2n^2 - 64a^4 + 126xy + 16a^2mn + 49y^2$.

49. $9a^4 + 5a^2x^2 + 25x^4$.

50. $a^4 + 4a^2b^2 + 16b^4$.

51. $625x^4 + 100x^2y^2 + 16y^4$.

6. **Reduction of Fractions.** Since the terms of the fraction a/b represent numbers, the rules for the manipulation of fractions are the same in algebra as in arithmetic. Some of the principles most frequently used will be set down on the next few pages for reference.

If both the numerator and the denominator of a fraction be multiplied or divided by the same number, the value of the fraction is not changed.

This principle is often used to reduce a fraction to its lowest terms, or to change it to a fraction having a given denominator.

Thus

$$\frac{12}{20} = \frac{12 \div 4}{20 \div 4} = \frac{3}{5},$$

and

$$\frac{4}{9} = \frac{4 \times 2}{9 \times 2} = \frac{8}{18}.$$

EXAMPLE 1. Reduce the fraction

$$\frac{15a^3bx^2}{20a^2b^3x}$$

to its lowest terms.

The factors which are common to the numerator and denominator are 5, a^2, b, and x. Dividing both terms by $5a^2bx$ we obtain

$$\frac{3ax}{4b^2}.$$

This is usually done by cancellation as in arithmetic. Thus

$$\frac{\overset{3}{\cancel{15}}\,\cancel{a^2}\,\cancel{b}\,\cancel{x}}{\underset{4}{\cancel{20}}\,\cancel{a^2}\,\cancel{b^2}\,\cancel{x}} = \frac{3ax}{4b^2}.$$

EXAMPLE 2. Reduce the fraction

$$\frac{8x^3 - 125}{2x^3 + x^2 - 15x}$$

to its lowest terms.

$$\frac{8x^3 - 125}{2x^3 + x^2 - 15x} = \frac{(2x - 5)(4x^2 + 10x + 25)}{x(2x^2 + x - 15)}$$

$$= \frac{\cancel{(2x - 5)}(4x^2 + 10x + 25)}{x\cancel{(2x - 5)}(x + 3)} = \frac{4x^2 + 10x + 25}{x^2 + 3x}.$$

EXAMPLE 3. Change the fraction

$$\frac{x}{4 - x}$$

to a fraction whose denominator is $16 - x^2$.

Multiply both numerator and denominator by $4 + x$. Thus

$$\frac{x}{4 - x} = \frac{x\cancel{(4 + x)}}{(4 - x)\cancel{(4 + x)}} = \frac{4x + x^2}{16 - x^2}.$$

EXERCISES

Reduce each of the following fractions to its lowest terms.

1. $\dfrac{35a^2b^3c}{21a^3bc^4}$.

2. $\dfrac{90x^7y^4}{36x^6y^3}$.

3. $\dfrac{a^2 + 3a + 2}{a^2 + 6a + 5}$.

4. $\dfrac{4a^2 - 25}{4a^2 - 20a + 25}$.

5. $\dfrac{25 - x^2}{x^2 - 11x + 30}$.

6. $\dfrac{a^2 + a - 12}{3a^2 - 13a + 12}$.

7. $\dfrac{15x^4y + 10x^3y^2}{6x^3y^4 + 4x^2y^5}$.

8. $\dfrac{x^4 - 8x}{x^3 - 2x^2}$.

9. $\dfrac{m^2 - 1}{m^6 - 1}$.

10. $\dfrac{m^6 - n^2}{m^6 + 2m^3n - 3n^2}$.

11. $\dfrac{a^2 - 9b^2 - c^2 + 6bc}{a^2 - 9b^2 + c^2 - 2ac}$.

12. $\dfrac{4a^3 + 13a^2b - 4ab^2 - 6b^3}{8a^2 + 14ab - 15b^2}$.

7. **Addition and Subtraction of Fractions.** Here again the method is the same as in arithmetic. If the fractions do not have the same denominator, they must first be changed (§ 6) to equivalent forms, having the same denominator. Then their sum, or difference, is equal to the sum, or difference, of their numerators, divided by this denominator.

Thus

$$\frac{3}{5} + \frac{1}{3} = \frac{9}{15} + \frac{5}{15} = \frac{9 + 5}{15} = \frac{14}{15}.$$

This same method is used when any number of fractions are added and subtracted.

The expression of lowest degree which is exactly divisible by each of the several denominators is usually taken as the common denominator of the fractions. This expression is known as the *least common multiple* of the denominators.

Example 1.

$$\frac{x}{3} + \frac{x}{4} = \frac{4x}{12} + \frac{3x}{12} = \frac{4x + 3x}{12} = \frac{7x}{12}.$$

Example 2.

$$\frac{3}{x - y} - \frac{x + 2y}{x^2 - y^2} = \frac{3(x + y)}{x^2 - y^2} - \frac{x + 2y}{x^2 - y^2}$$

$$= \frac{(3x + 3y) - (x + 2y)}{x^2 - y^2} = \frac{3x + 3y - x - 2y}{x^2 - y^2} = \frac{2x + y}{x^2 - y^2}.$$

EXAMPLE 3.

$$\frac{2x+3}{6x} - \frac{3x^2+1}{12x^2} - \frac{3x^3-4}{36x^3}$$

$$= \frac{(2x+3)6x^2}{36x^3} - \frac{(3x^2+1)3x}{36x^3} - \frac{3x^3-4}{36x^3}$$

$$= \frac{(12x^3+18x^2) - (9x^3+3x) - (3x^3-4)}{36x^3}$$

$$= \frac{12x^3+18x^2-9x^3-3x-3x^3+4}{36x^3} = \frac{18x^2-3x+4}{36x^3}.$$

EXERCISES

Reduce each of the following expressions to a single fraction.

1. $\frac{7a}{10} - \frac{6a}{15} + \frac{3a}{5}$.

2. $\frac{2a}{3b} + \frac{11}{2ab^2} - \frac{5b}{a}$.

3. $\frac{3a-2}{5} - \frac{a+7}{2} + 4$.

4. $\frac{x-6}{10} + \frac{x-5}{15} - \frac{x+2}{20}$.

5. $\frac{a-2b}{2a} - \frac{a-5b}{4a} + \frac{a+7b}{8a}$.

6. $\frac{a^2-bc}{bc} - \frac{ac-b^2}{ac} - \frac{ab-c^2}{ab}$.

7. $\frac{2a^2-b^2}{a^2} - \frac{b^2-c^2}{b^2} - \frac{c^2-a^2}{c^2}$.

8. $\frac{a-b-c}{4} - \frac{5b-3c-a}{7} - \frac{5c-10b+6a}{14}$.

9. $\frac{3}{x-6} - \frac{1}{x+2}$.

10. $\frac{2x-3a}{x-2a} - \frac{2x-a}{x-a}$.

11. $\frac{2x+3y}{2x-3y} - \frac{2x-3y}{2x+3y}$.

12. $\frac{x-1}{x-2} - \frac{x-3}{x-1}$.

13. $a + x + \frac{x^2}{a-x}$.

14. $\frac{3}{4x+y} - \frac{2}{4x-y}$.

15. $\frac{2a}{a^2-1} - \frac{1}{a+1}$.

16. $\frac{1}{1+a} + \frac{1}{1-a} - \frac{2a}{1-a^2}$.

17. $\frac{2}{a^2-7a} + \frac{3}{a-7} - \frac{3}{a}$.

18. $\frac{a-2b}{a^2-ab} + \frac{a+b}{a^2-2ab+b^2}$.

19. $\frac{1}{a^2-9a+20} + \frac{1}{a^2-11a+30}$.

20. $\dfrac{a^2+ab+b^2}{a+b}+\dfrac{a^2-ab+b^2}{a-b}$. 21. $\dfrac{a}{a-b}+\dfrac{2ab}{a^2-b^2}-\dfrac{2b}{a+b}$.

22. $\dfrac{2x-3}{1-4x^2}+\dfrac{3}{1-2x}+\dfrac{2}{x}$. 23. $1+2a+4a^2+\dfrac{a^3+1}{2a-1}$.

24. $\dfrac{2r-1}{r^2+2r-3}-\dfrac{r-2}{r^2-1}-\dfrac{1}{r+3}$.

25. $\dfrac{2}{a^2-3a+2}-\dfrac{3}{a^2-5a+6}+\dfrac{4}{a^2-4a+3}$.

26. $\dfrac{5}{a-2}+\dfrac{7}{a-1}-\dfrac{5}{a+2}-\dfrac{7}{a+1}$.

27. $\dfrac{a+1}{2a^2+5a+2}-\dfrac{2a+1}{4a^2+5a-6}+\dfrac{3}{8a^2-2a-3}$.

28. $\dfrac{1}{8-8x}-\dfrac{1}{8+8x}+\dfrac{x}{4+4x^2}-\dfrac{x}{2+2x^4}$.

29. $\dfrac{4ab}{a^3+8b^3}+\dfrac{1}{a+2b}+\dfrac{a-2b}{a^2-2ab+4b^2}$.

30. $\dfrac{1}{6x-18}-\dfrac{1}{6x+18}-\dfrac{1}{x^2+9}+\dfrac{18}{x^4+81}$.

31. $\dfrac{x^2}{(x-y)(x-z)}+\dfrac{y^2}{(y-x)(y-z)}+\dfrac{z^2}{(z-x)(z-y)}$.

32. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-a)(b-c)}+\dfrac{c}{(c-a)(c-b)}$.

33. $\dfrac{1+a}{(a-b)(a-c)}+\dfrac{1+b}{(b-c)(b-a)}+\dfrac{1+c}{(c-a)(c-b)}$.

8. **Multiplication and Division of Fractions.** The product of two or more fractions is the product of the numerators divided by the product of the denominators. Common factors in the numerators and denominators should be divided out before performing the multiplication.

Thus

$$\frac{2}{3}\times\frac{9}{7}\times\frac{5}{2}=\frac{\cancel{2}\times\overset{3}{\cancel{9}}\times 5}{\cancel{3}\times 7\times\cancel{2}}=\frac{15}{7}.$$

In order to divide one fraction by another, invert the divisor and multiply.

Thus

$$\frac{3}{5}\div\frac{2}{3}=\frac{3}{5}\times\frac{3}{2}=\frac{9}{10}.$$

Mixed expressions should be written in fractional form before applying these rules.

EXAMPLE 1.

$$\frac{ab}{a^2-b^2}\times\frac{a-b}{a}=\frac{ab(a-b)}{(a+b)(a-b)a}=\frac{b}{a+b}.$$

EXAMPLE 2.

$$\frac{25x^2-16y^2}{x^2-y^2}\div\frac{10x^2-3xy-4y^2}{x^2-xy}$$

$$=\frac{(5x+4y)(5x-4y)}{(x+y)(x-y)}\times\frac{x(x-y)}{(5x-4y)(2x+y)}$$

$$=\frac{(5x+4y)x}{(x+y)(2x+y)}=\frac{5x^2+4xy}{2x^2+3xy+y^2}.$$

EXERCISES

Perform the operations indicated:

1. $\dfrac{5a^2b}{3c^5}\times\dfrac{6c^2}{10ab^2}$.

2. $\dfrac{30x^2y}{8z^3}\times\dfrac{16yz^2}{45x}$.

3. $\dfrac{42p^2q}{25r^3}\div\dfrac{6p}{5r}$.

4. $\dfrac{14y^3z}{15x^6}\times\dfrac{5xz^3}{12y^5}\times\dfrac{6x^2y}{7z^4}$.

5. $\dfrac{a^2-3a+9}{2a-6}\times\dfrac{a^2-9}{a^3+27}$.

6. $\dfrac{16x^2-9a^2}{x^2-4}\div\dfrac{4x-3a}{x-2}$.

7. $\dfrac{b^4-16}{b^3-1}\div\dfrac{2-b-b^2}{1+b+b^2}$.

8. $\dfrac{2a^2-4a+8}{1}\div\dfrac{a^4+8a}{3a}$.

9. $\dfrac{m^2 - 6m + 8}{m^2 + 2m + 1} \div \dfrac{m - 4}{m + 1}$.

10. $\dfrac{a^2 - 3ab}{a^3 - b^3} \div \dfrac{a^2 - 10ab + 21b^2}{a^2 + ab + b^2}$.

11. $\left(x + \dfrac{xy}{x - y}\right)\left(y - \dfrac{xy}{x + y}\right)$.

12. $\dfrac{x^3 + 5x^2 - 36x}{x^2 - 7x - 144} \times \dfrac{x^2 - 19x + 48}{x^3 - 2x^2 - 8x}$.

13. $\dfrac{1 - a}{a^3 + a^4 - a^5} \div \dfrac{1 - a^3}{a^5 - a^3 - 2a^2 - a}$.

14. $\left(\dfrac{x^2 + 1}{2x - 1} - \dfrac{x}{2}\right)\left(\dfrac{1 - 2x}{x + 2}\right)$.

15. $\dfrac{a^3 + a^2b + ab^2 + b^3}{a^3 - 3a^2b + 3ab^2 - b^3} \div \dfrac{a^3 + b^3}{a^3 - b^3}$.

16. $\dfrac{m^2 + 2mn + n^2 - p^2}{m^2 - 2mn + n^2 - p^2} \times \dfrac{m - n + p}{m + n - p}$.

17. $\dfrac{5a^4 - 5a^3}{7a^2 - 56a - 63} \div \dfrac{a^4 - 9a^3 + 8a^2}{14a^2 + 14a - 1260}$.

18. $\dfrac{a^4 - b^4}{(a + b)^2} \times \dfrac{a^2 - b^2}{a^2 + b^2} \times \dfrac{a + b}{(a - b)^2}$.

19. $\left(x^2 - 3x + 2 - \dfrac{12}{x + 3}\right)\left(3x + 6 - \dfrac{2x^2}{x - 3}\right)$.

20. $\left(\dfrac{2}{a} - \dfrac{1}{a + b} + \dfrac{1}{a - b}\right) \div \left(\dfrac{a + b}{a - b} - \dfrac{a - b}{a + b}\right)$.

21. $\dfrac{(a + b)^2 - c^2}{a^2 + ab - ac} \times \dfrac{a}{(a + c)^2 - b^2} \times \dfrac{(a - b)^2 - c^2}{ab - b^2 - bc}$.

22. $\dfrac{2ac - 6ad - bc + 3bd}{3ac + ad - 6bc - 2bd} \times \dfrac{a^2 - 7ab + 10b^2}{10a^2 - 3ab - b^2}$.

9. **The Linear Equation.** Any statement of equality between numbers or symbols is called an ***equation.*** If an equation holds for all values of the symbols involved for which the expressions can be evaluated, it is said to be an ***identical equation,*** or an ***identity.***

Thus

$$3 + 4 = 7, \qquad (x - 1)^2 = x^2 - 2x + 1,$$

and

$$a^2 - b^2 = (a + b)(a - b)$$

are identities.

An ***equation of condition,*** or an ***equation,*** is an equation involving one or more symbols, called *unknown numbers,* which is satisfied only by particular values, or sets of values, of these symbols.

Thus $x + 3 = 7$ is true for $x = 4$ and for no other value of x. Also $x - y = 3$ is true for many sets of values of x and y but not for every set. For example, it is true for $x = 5$, $y = 2$; $x = 7$, $y = 4$; and $x = -5$, $y = -8$; but not for $x = 6$, $y = 1$.

Any value, or set of values, which satisfies an equation is called a ***solution*** of the equation. If the equation contains only one unknown number, its solutions are called ***roots*** of the equation.

An equation which, when cleared of fractions, is of the first degree in the unknown numbers, is said to be a ***linear equation.*** The following examples illustrate the method of solving a linear equation in one unknown quantity.

Example 1. Solve the equation

$$\frac{8x - 11}{9} - \frac{7x + 4}{12} = \frac{3x - 8}{8}.$$

Multiplying both members of the equation by 72, in order to clear of fractions, gives

$$8(8x - 11) - 6(7x + 4) = 9(3x - 8),$$

or

$$64x - 88 - 42x - 24 = 27x - 72.$$

Transposing and collecting terms, we obtain

$$-5x = 40.$$

Therefore

$$x = -8.$$

CHECK. Substituting -8 for x in the original equation gives

$$\frac{-64-11}{9}-\frac{-56+4}{12}=\frac{-24-8}{8},$$

or

$$\frac{-75}{9}-\frac{-52}{12}=\frac{-32}{8},$$

or

$$\frac{-25}{3}+\frac{13}{3}=-4,$$

or

$$\frac{-12}{3}=-4.$$

NOTE. This method of checking the solution is of no value unless the substitution be made in the *original* equation. Why?

EXAMPLE 2. Solve the equation

$$\frac{x-a}{x-b}=\frac{b}{a}.$$

Multiplying both members of the equation by $a(x-b)$ gives

$$a(x-a)=b(x-b),$$

or

$$ax-a^2=bx-b^2,$$

or

$$x(a-b)=a^2-b^2.$$

Since we may divide both members of an equation by any number we please, except zero, we must assume at this point that $b \neq a$. Division by $a-b$ then gives

$$x=a+b.$$

If $b=a$, the original equation becomes

$$\frac{x-a}{x-a}=\frac{a}{a},$$

which is true for every value of x.

NOTE. Before dividing an equation by any literal factor the student should be sure that this factor is not equal to zero,

otherwise serious errors may result. Where is the error in the following argument?

Suppose

$$a = x.$$

Then

$$ax = x^2,$$

and

$$ax - a^2 = x^2 - a^2,$$

or

$$(x - a)a = (x - a)(x + a).$$

Dividing by $x - a$, we find

$$a = x + a.$$

Then, since $x = a$, we have

$$a = 2a,$$

or

$$1 = 2.$$

EXERCISES

Solve each of the following equations.

1. $\dfrac{2x + 3}{11} + \dfrac{x - 1}{3} = 2.$

2. $\dfrac{6x + 7}{12} = \dfrac{3x - 4}{4x - 3} + \dfrac{x}{2}.$

3. $\dfrac{3a - 4}{a + 2} = \dfrac{6a - 11}{2a + 1}.$

4. $\dfrac{2x + 13}{3} - \dfrac{6 - x}{4} = 1.$

5. $.04x = .1x + 2.4.$

6. $\dfrac{2x}{5} + \dfrac{2(1 - x)}{7x - 6} + \dfrac{7}{15} = \dfrac{2x + 1}{5}.$

7. $\dfrac{8x + 57}{12} = \dfrac{2x - 15}{x + 8} + \dfrac{2x + 16}{3}.$

8. $(5x - 2)(3x - 4) = (3x + 5)(5x - 6).$

9. $\dfrac{1}{a} - \dfrac{1}{x} = \dfrac{1}{x} - \dfrac{1}{b}.$

10. $.12(2x + .5) - .2(1.5x - 2) = .4.$

11. $(x - a)(b - c) - (x - c)(b - a) = 0.$

12. $\dfrac{ax - b}{c} - \dfrac{bx + c}{a} = abc.$

13. $\frac{17a-5}{3}-\frac{10a+2}{4}=\frac{5a+7}{2}-5.$

14. $\frac{(a+1)x}{b}+\frac{(b+1)x}{a}+\frac{2ab}{a+b}=a+b+1.$

15. $\frac{1}{2}\left(x-\frac{a}{3}\right)-\frac{1}{3}\left(x-\frac{a}{4}\right)+\frac{1}{4}\left(x-\frac{a}{5}\right)=0.$

16. $5(2a+7)(a-2)-6(a+4)^2=5+(2a+3)^2.$

17. $\frac{x+5}{x+4}-\frac{x-6}{x-7}=\frac{x-4}{x-5}-\frac{x-15}{x-16}.$

18. $\frac{x+3a}{x+a}+\frac{x+2a}{x+5a}=\frac{x+a}{x+3a}+\frac{x+6a}{x+5a}.$

19. What number is as much under $11\frac{1}{4}$ as it is over $6\frac{1}{2}$?

20. Separate 102 into two parts such that if the first part is divided by the second part the quotient is 3 and the remainder is 6.

21. There are three numbers such that the second is 6 more than 4 times the first, and the third is 3 less than 6 times the first. If one-third of the third is subtracted from the second, the remainder is 17. Find the numbers.

22. A freight train runs 17 miles an hour less than a passenger train. It runs 136 miles in the same time that the passenger train runs 204 miles. Find the rate of each.

23. A man, traveling at the rate of 20 miles a day, has four days' start of another man who travels at the rate of 25 miles a day in the same direction. After how many days will the second man overtake the first?

24. A man invests part of \$5,600.00 at 6%, and the remainder at 5%. The investment at 6% yields annually twice as much as the investment at 5%. Find the sum invested at 5%.

25. A man has two-thirds of his money invested at 4%, one-fourth at 3%, and the remainder at 2%. If his income is \$430.00, what is his total investment?

26. A is 24 years old; B is 52 years old. When will A's age be half of B's?

27. A river flows at the rate of 4 miles an hour. A man who rows at the rate of 6 miles an hour in still water wishes to row 4 miles up the stream. How long will it take him?

28. In traveling a certain distance a train that goes 45 miles an hour takes 2 hours less than a train that goes 40 miles an hour. Find the distance.

29. A can do a piece of work in one-half the time that B can; B can do it in two-thirds of the time that C can; all together they can do it in 6 days. Find the time it would take each of them alone.

30. A tank can be filled with one pipe in 30 minutes, by a second pipe in 40 minutes, by a third in 50 minutes. How long will it take to fill it with them all running together?

10. Simultaneous Linear Equations. A set of two or more linear equations, each of which is supposed to be satisfied by the same values of the unknown numbers, is called a system of ***simultaneous linear equations.*** A set of values of the unknown numbers which satisfies each of the equations is called a ***solution*** of the system of equations. Methods of solving such a set of equations are illustrated below.

EXAMPLE 1. Solve the equations

$$\begin{cases} 2x + y = 10, & (1) \\ 3x - 2y = 1. & (2) \end{cases}$$

First solution. Multiply (1) by 3 and (2) by 2. The two equations then become

$$6x + 3y = 30,$$

$$6x - 4y = 2.$$

Subtraction gives $7y = 28$, or $y = 4$.
Substituting 4 for y in (2) gives

$$3x - 8 = 1, \quad \text{or} \quad 3x = 9, \quad \text{or} \quad x = 3.$$

Hence the required set of values of x and y is $x = 3$, $y = 4$.

NOTE. It is evident that the symbols which are used for the unknown numbers are of no significance. For example, the

system of equations

$$\begin{cases} 2m + n = 10, \\ 3m - 2n = 1, \end{cases}$$

is solved as above and the solution is $m = 3$, $n = 4$.

Second solution. A glance at equations (1) and (2) shows that if they be multiplied by 3 and -2, respectively, and added, the letter x will vanish. Let us merely indicate these multipliers as follows:

$$\begin{array}{rl} 2x + y = 10 & \Big| \quad 3 \\ 3x - 2y = 1 & \Big| -2 \end{array}$$

and perform the multiplication and addition mentally. We obtain

$$(3 + 4)y = 30 - 2, \quad \text{or} \quad 7y = 28, \quad \text{or} \quad y = 4.$$

Likewise, after writing the multipliers which eliminate y by addition,

$$\begin{array}{rl} 2x + y = 10 & \Big| \, 2 \\ 3x - 2y = 1 & \Big| \, 1 \end{array}$$

we find

$$(4 + 3)x = 20 + 1, \quad \text{or} \quad 7x = 21, \quad \text{or} \quad x = 3.$$

This method of solution is preferable for sets of equations of this type.

Third solution. Solving (1) for y gives

$$y = 10 - 2x.$$

Substituting $10 - 2x$ for y in (2), we get

$$3x - 2(10 - 2x) = 1,$$

or

$$3x - 20 + 4x = 1.$$

Therefore

$$7x = 21, \quad \text{or} \quad x = 3.$$

Then

$$y = 10 - 2x = 10 - 6 = 4.$$

EXAMPLE 2. Solve the system of equations

$$\begin{cases} 3ax + 2by = 5ab, \\ bx - ay = b^2 - a^2. \end{cases}$$

As in the second solution above, we write

$$\begin{array}{rr|r} 3ax + 2by = 5ab & & b \\ bx - ay = b^2 - a^2 & & -3a \end{array}$$

Hence

$$\begin{aligned}(2b^2 + 3a^2)y &= 5ab^2 - 3ab^2 + 3a^3 \\ &= a(2b^2 + 3a^2),\end{aligned}$$

or

$$y = a.$$

Also

$$\begin{array}{rr|r} 3ax + 2by = 5ab & & a \\ bx - ay = b^2 - a^2 & & 2b \end{array}$$

Hence

$$\begin{aligned}(3a^2 + 2b^2)x &= 5a^2b + 2b^3 - 2a^2b \\ &= b(3a^2 + 2b^2),\end{aligned}$$

or

$$x = b.$$

Hence the required set of values of x and y is $x = b$, $y = a$.

Example 3. Solve the system of equations

$$\begin{cases} 3x - 2y + z = 2, \\ 2x + 3y - 4z = -4, \\ x - 5y + 2z = -3. \end{cases}$$

Multiply the first equation by 4 and the second by 1, and add; then multiply the first equation by 2 and the third by -1, and add. This reduces the system to a system of two equations in two unknowns. These multiplications may be indicated as follows:

$$\begin{array}{rr|r|r} 3x - 2y + z = 2 & & 4 & \\ 2x + 3y - 4z = -4 & & 1 & \\ x - 5y + 2z = -3 & & & -1 \end{array}$$

The result is

$$\begin{array}{rr|r|r} 14x - 5y = 4 & & 1 & -5 \\ 5x + y = 7 & & 5 & 14 \end{array}$$

Multiplying now by 1 and 5, and -5 and 14, and adding, we find

$$39x = 39, \quad \text{or} \quad x = 1,$$

and

$$39y = 78, \quad \text{or} \quad y = 2.$$

Substituting these values of x and y in one of the given equations, we have $z = 3$. Hence the solution is $x = 1$, $y = 2$, $z = 3$.

In general, if the equations are linear and the number of the equations is equal to the number of the unknowns, there will exist one, and only one, solution. However, not every set of linear equations has a solution. For example, there is no set of values of x and y that satisfies the equations

$$2x + 3y = 6,$$
$$4x + 6y = 10.$$

It is evident that any set of values of x and y which makes $2x + 3y$ equal to 6 must make $4x + 6y$ equal to 12 and not 10. Hence this system of equations has no solution. In this case the equations are not *consistent.*

On the other hand, a system of equations may have infinitely many solutions. For example, there are many sets of values of x and y that satisfy both the equations

$$2x + 3y = 6,$$
$$8x + 12y = 24.$$

It is evident that any set of values of x and y which satisfies the first equation must also satisfy the second equation. Hence the system of equations is satisfied by infinitely many pairs of values of x and y. In this case the equations are not *independent;* that is, one of the equations may be derived from the other by multiplying it by a constant.

EXERCISES

Solve each of the following systems of linear equations.

1. $\begin{cases} 3x + y = 31, \\ 5x - 2y = 15. \end{cases}$

2. $\begin{cases} 8b + 2a + 1 = 0, \\ 4b - 10a - 7 = 0. \end{cases}$

3. $\begin{cases} 12x - 14y + 4 = 0, \\ 16x - 42y + 17 = 0. \end{cases}$

4. $\begin{cases} 5x + 4y = 9a + b, \\ 7x - 6y = a + 13b. \end{cases}$

5. $\begin{cases} \frac{5}{a}+\frac{3}{b}=30, \\ \frac{9}{a}-\frac{5}{b}=2. \end{cases}$

8. $\begin{cases} \frac{2ax-by}{a}=b, \\ \frac{x+by}{3a+2}=b. \end{cases}$

6. $\begin{cases} \frac{m}{3}-\frac{n}{4}=\frac{1}{12}, \\ \frac{m}{2}-\frac{3n}{16}=\frac{1}{2}. \end{cases}$

9. $\begin{cases} \frac{x}{a+b}-\frac{y}{a-b}=\frac{2}{a+b}, \\ \frac{x}{a+b}+\frac{y}{a-b}=\frac{2}{a-b}. \end{cases}$

7. $\begin{cases} \frac{3}{a-4}+\frac{4}{b-1}=3, \\ \frac{9}{a-4}-\frac{2}{b-1}=2. \end{cases}$

10. $\begin{cases} \frac{2m+n+6}{m-2n-3}+\frac{2}{7}=0, \\ 5m+2n+7=0. \end{cases}$

11. $\begin{cases} \frac{3}{x-1}-\frac{4}{y-2}=0, \\ \frac{7}{x+y-3}+\frac{1}{x-y+1}=0. \end{cases}$

12. $\begin{cases} \frac{3x+7y+1}{5}-\frac{2x-3y+8}{3}=2, \\ \frac{5x-7y+10}{3}-\frac{3x+2y+6}{5}=2. \end{cases}$

13. $\begin{cases} 6x+2y-5z=13, \\ 3x+3y-2z=13, \\ 7x+5y-3z=26. \end{cases}$

14. $\begin{cases} ax+by=(a+b)c, \\ by+cz=(c+a)b, \\ cz+ax=(b+c)a. \end{cases}$

15. $\begin{cases} 3x+4y+2z=14, \\ x-3y+5z=4, \\ 2x+7y-3z=6. \end{cases}$

16. $\begin{cases} x+y+z=(a+b+c)^2, \\ ay+bz+cx=3(ab^2+bc^2+ca^2), \\ ax+by+cz=a^3+b^3+c^3+6abc. \end{cases}$

17. $\begin{cases} \frac{1}{x} + \frac{2}{y} + \frac{2}{z} = 16, \\ \frac{2}{x} + \frac{1}{y} + \frac{2}{z} = 15, \\ \frac{2}{x} + \frac{2}{y} + \frac{1}{z} = 14. \end{cases}$

18. If the larger of two numbers be divided by the smaller, the quotient is 4 and the remainder 1; but if the smaller be divided by the larger, the quotient is 0.24. Find the numbers.

19. A boat travels 20 miles an hour down a river. It takes the boat twice as long to go a certain distance up the river as to go the same distance down the river. Find the rate at which the river flows and the rate of the boat in still water.

20. If 1 be subtracted from the numerator and denominator of a certain fraction, the resulting fraction will be equal to $\frac{1}{2}$; but if 1 be added to the numerator and denominator of the same fraction, the resulting fraction will be equal to $\frac{3}{5}$. What is the fraction?

21. A man has 12 coins, including only dimes, quarter-dollars, and half-dollars, to the value of $3.80. If he replaces the quarters by dimes, and the dimes by quarters, the value of the coins will be $3.65. How many has he of each?

22. A man invests $40,000.00, a part at 5% and a part at 6%. If his income is $2,150.00 per year, how is his money divided?

23. Find two numbers such that seven times the first plus four times the second equals 74, and eight times the first minus three times the second equals 24.

24. Two numbers are written with the same two digits; the sum of the numbers is 132, and the difference of the digits is 4. Find the numbers.

25. The sum of the three digits of a number is 15. The digit in tens' place exceeds that in hundreds' place by 2; and if 396 be added to the number, the digits will be reversed. Find the number.

26. The sum of the three digits of a number is 14, and the sum of the first and third digits exceeds the second digit by 4. If the

hundreds' and units' digits are interchanged, the number is diminished by 693. Find the number.

27. Two pipes empty a reservoir in 2 hours and 24 minutes. When one pipe is used to fill and one to empty the reservoir, it becomes full in 12 hours. How long would it take each pipe separately to empty the reservoir?

28. A and B can do a piece of work in 18 days, B and C in 24 days, and C and A in 36 days. How long will it require each man, working alone, to do it, and how long will it require all working together?

29. A man has two sons, one 4 years older than the other. After 2 years the father's age will be twice the joint ages of his sons; and 6 years ago his age was six times the joint ages of his sons. How old is the man and each of his sons?

30. A farmer has enough feed for his horses to last a certain number of days. If he were to sell 75 horses, his feed would last 20 days longer. If he were to buy 100 horses, his feed would last 15 days less. How many horses has he and how long will his feed last?

11. The Solution of a System of Linear Equations by Determinants. If the system of equations *

$$\begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2, \end{cases}$$

be solved by any of the methods of § 10, the solution will be

$$x = (c_1b_2 - c_2b_1)/(a_1b_2 - a_2b_1),$$
$$y = (a_1c_2 - a_2c_1)/(a_1b_2 - a_2b_1).$$

This solution may be written in another form by making use of the symbol

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix},$$

* In algebra subscripts are frequently added to letters for purposes of symmetry. The symbols a_1, a_2, b_1, etc. (read "a one, a two, b one," etc.) are to be treated as general algebraic symbols (letters).

which stands for $ad - bc$, and is called a ***determinant*** of the *second order*. Thus the solution of the above system of equations is

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

Thus the value of each of the unknowns can be expressed as the quotient of two determinants. The denominator in each case is the determinant of the coefficients, and the numerator is obtained from the denominator by replacing the coefficients of the unknown by the constant terms.

It is evident that if the common denominator of the two fractions is equal to zero, no solution exists. In other words, there exists a unique solution of this system of equations when and only when

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

The student should now apply this test to each of the two systems discussed near the end of § 10.

In order that these results may be extended to a system of three linear equations in three unknowns we make use of another symbol

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

which stands for

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3,$$

and is called a determinant of the *third order*.

There is a very simple method for the evaluation of any determinant of the third order. The terms are obtained by

multiplying together the elements connected by lines in the following scheme:

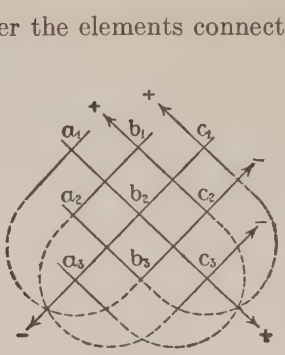


FIG. 1

EXAMPLE.

$$\begin{vmatrix} 2 & 4 & -5 \\ 1 & -2 & 3 \\ 4 & -1 & 2 \end{vmatrix} = 2 \times (-2) \times 2 + 1 \times (-1) \times (-5) + 4 \times 3 \times 4 - 4 \times (-2) \times (-5) - (-1) \times 3 \times 2 - 2 \times 4 \times 1$$
$$= -8 + 5 + 48 - 40 + 6 - 8 = 3.$$

It can be shown that the solution of the system of equations

$$\begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2, \\ a_3x + b_3y + c_3z = d_3, \end{cases}$$

may be written in the form

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

although the proof will be omitted here.

As in the preceding example the denominator in each case is the determinant of the coefficients and the numerator is obtained from the denominator by replacing the coefficients of the unknown by the constant terms.

It is evident that a unique solution of this system of equations exists when and only when

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0.$$

This method of solving systems of equations by determinants is a general one and may be applied to systems containing more than three equations. Such systems will be discussed later (§ 130). It has one advantage over other methods in that the value of any one of the unknowns may be found independent of the others.

EXAMPLE. Solve by determinants the system of equations

$$\begin{cases} x + 2y + 3z = 4, \\ 3x + 5y + z = 18, \\ 4x + y + 2z = 12. \end{cases}$$

We have

$$x = \frac{\begin{vmatrix} 4 & 2 & 3 \\ 18 & 5 & 1 \\ 12 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 3 & 5 & 1 \\ 4 & 1 & 2 \end{vmatrix}} = \frac{40 + 54 + 24 - 180 - 72 - 4}{10 + 9 + 8 - 60 - 12 - 1} = \frac{-138}{-46} = 3.$$

In like manner we find $y = 2$ and $z = -1$. The solution of the system is, therefore, $x = 3$, $y = 2$, $z = -1$.

EXERCISES

Solve the following systems of equations, using determinants:

1. $\begin{cases} 6a + 9b = 54, \\ 5a - 3b = 3. \end{cases}$

2. $\begin{cases} 6y - 12x + 29 = 0, \\ 6y - 6x + 17 = 0. \end{cases}$

3. $\begin{cases} 3x + 2y - z = 7, \\ x + y - z = 1, \\ x + 2y + 3z = 15. \end{cases}$

4. $\begin{cases} x + y + z = 1, \\ 8x - 2y + 5z = 3, \\ 9x - y + 6z = 4. \end{cases}$

5. $\begin{cases} 3x + 5y = 35, \\ 3y + 5z = 27, \\ 3z + 5x = 34. \end{cases}$

6. $\begin{cases} 3x + 2y + 4z = 19, \\ 2x + 5y + 3z = 21, \\ 3x - y + z = 4. \end{cases}$

7. $\begin{cases} 2x - y + 6z = 0, \\ 3x + 2y - z = 0, \\ 7x + 28y - 30z = 0. \end{cases}$

8. $\begin{cases} p + q + r = 7, \\ p + 2q + 3r = 10, \\ 2p + 3q + 6r = 15. \end{cases}$

9. $\begin{cases} 2a - 3b + 4c = 1, \\ a + b - 5c = 2, \\ 3a - 2b - 3c = 1. \end{cases}$

10. $\begin{cases} x + ay + a^2z + a^3 = 0, \\ x + by + b^2z + b^3 = 0, \\ x + cy + c^2z + c^3 = 0. \end{cases}$

12. **The Binomial Theorem.** Repeated multiplication of the binomial $a + x$ by itself gives

$$(a + x)^2 = a^2 + 2ax + x^2,$$
$$(a + x)^3 = a^3 + 3a^2x + 3ax^2 + x^3,$$
$$(a + x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4,$$
$$(a + x)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5, \text{ etc.}$$

We can thus find by successive multiplications the expanded form of $(a + x)^n$, where n is any positive integer. This method would become rather laborious if we should undertake to expand some power of $a + x$ such as $(a + x)^{12}$. However, a comparison of these results discloses a formula by means of which the terms in each of the expanded forms may be obtained.

If n is a positive integer, we observe that the expansion of $(a + x)^n$ has the following properties:

1. The exponent of a in the first term is the same as the exponent of the binomial, and it decreases by 1 in each succeeding term.

2. The first term does not contain x. The exponent of x in the second term is 1 and it increases by one in each succeeding term.

3. The coefficient of the first term is 1 and the coefficient of the second term is the exponent of the binomial.

4. If the coefficient of any term be multiplied by the exponent of a in that term, and the product be divided by the number of the term, the quotient is the coefficient of the next term.

5. The total number of terms is one more than the exponent of the binomial.

6. The coefficients of terms equidistant from the ends of the expansion are equal.

7. The sum of the exponents in any term is equal to the exponent of the binomial.

These results may now be written in the symbolic language of algebra as follows:

$$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3 + \cdots + nax^{n-1} + x^n,$$

where n is any positive integer. The dots indicate that the process is to be carried on until all of the $n + 1$ terms have been found.

This formula, which is known as the ***Binomial Theorem,*** enables one to write down at once the expanded form of any binomial raised to any positive integral power. Although this formula is true for all positive integral values of n, and also for certain other values, it should be observed

that we have not proved it. We have merely verified it for certain small positive integral values of n. The general proof will be deferred until a later chapter (§ 53).

EXAMPLE 1. Expand $(2 - 3b)^4$.

Here $a = 2$, $x = -3b$, and $n = 4$. The binomial formula gives

$$(2 - 3b)^4 = [2 + (-3b)]^4 = 2^4 + 4 \cdot 2^3(-3b) + \frac{4 \cdot 3}{1 \cdot 2} \cdot 2^2(-3b)^2 + \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} \cdot 2(-3b)^3 + \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4}(-3b)^4,$$

$$= 2^4 + 4 \cdot 2^3(-3b) + 6 \cdot 2^2(-3b)^2 + 4 \cdot 2(-3b)^3 + (-3b)^4.$$

Thus

$$(2 - 3b)^4 = 16 - 96b + 216b^2 - 216b^3 + 81b^4.$$

EXAMPLE 2. Find the value of 21^3.

$$21^3 = (20 + 1)^3 = 20^3 + 3 \cdot 20^2 \cdot 1 + 3 \cdot 20 \cdot 1^2 + 1^3$$
$$= 8000 + 1200 + 60 + 1 = 9261.$$

EXERCISES

Expand:

1. $(a + b)^8$.
2. $(a - b)^7$.
3. $(2a + x)^5$.
4. $(a - 2b)^5$.
5. $(p + 2q)^5$.
6. $(c - 5d)^4$.
7. $(x + 3y^2)^6$.
8. $(2 + 3x^2)^5$.
9. $(x^3 - y^2)^6$.
10. $(x - y^2)^7$.
11. $(xy - 1)^8$.
12. $(3m^2 + 2n)^6$.
13. $\left(a^2 - \frac{3}{b}\right)^4$.
14. $\left(4a + \frac{b}{2}\right)^3$.
15. $\left(\frac{2}{a} - \frac{a^2}{5}\right)^6$.
16. $\left(\frac{2a^2}{b} + \frac{b^2}{3a}\right)^5$.
17. $\left(4a - \frac{b}{3}\right)^4$.
18. $\left(\frac{x}{a} - \frac{a}{x}\right)^6$.
19. $\left(2x - \frac{a}{2}\right)^6$.

20. $\left(6a - \frac{b}{2}\right)^3$.

21. $\left(\frac{5a}{2} - \frac{4b}{3}\right)^4$.

22. $\left(\frac{x^2}{2y} + \frac{2y^2}{x}\right)^5$.

23. $\left(\frac{3x^3}{2a^2} - \frac{4a}{x^2}\right)^4$.

Making use of the binomial expansion, compute the value of each of the following:

24. 32^3.

25. 29^2.

26. 101^2.

27. 11^4.

28. 199^2.

29. 35^2.

30. Put $a = 1$ and $x = 1$ in the binomial expansion and show that the sum of the coefficients in that expansion is 2^n.

31. Put $a = 1$ and $x = -1$ in the binomial expansion and show that the sum of the coefficients of the odd terms is equal to the sum of the coefficients of the even terms.

32. Make a list of the coefficients of each power of a binomial from the first to the eighth and verify in each case the results of exercises 30 and 31.

13. The General Term of $(a + x)^n$. A study of the expansion of $(a + x)^n$ term by term reveals the following facts.

In the third, or $(2 + 1)$st, term there are *two* factors in the numerator and *two* in the denominator of the coefficient; the exponent of x is *two* and that of a is *two* less than n.

In the fourth, or $(3 + 1)$st, term there are *three* factors in the numerator and *three* in the denominator of the coefficient; the exponent of x is *three* and that of a is *three* less than n.

Similar statements may be made of any term, such as the $(r + 1)$st. The formula for the $(r + 1)$st term is, therefore,

$$(r+1)\text{st term} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdot \cdots r}\cdot a^{n-r}x^r.$$

This formula enables one to write any required term of the expansion $(a + x)^n$ without considering the other terms.

EXAMPLE. Find the sixth term of $(m - 3n)^9$.

Here $a = m$, $x = -3n$, $n = 9$, $r = 5$. Hence the sixth term is

$$\frac{9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot m^4(-3n)^5 = 126m^4(-3n)^5 = -30{,}618m^4n^5.$$

EXERCISES

In each of the following cases, find the term specified:

1. $(a - x)^7$, 4th term.
2. $(2a + x)^9$, 8th term.
3. $\left(\frac{a}{2} - \frac{2b}{3}\right)^{12}$, 5th term.
4. $\left(\frac{3a^2}{4} - \frac{x^2}{3}\right)^6$, 3d term.
5. $(a - 3x^2)^8$, 6th term.
6. $(m^3 + 2n^2)^{11}$, 5th term.
7. $\left(\frac{x^2}{a^2} - \frac{a^2}{x^2}\right)^{10}$, 7th term.
8. $\left(p^2 - \frac{q}{2}\right)^{16}$, 4th term.
9. $\left(a^2 + \frac{2b^2}{3}\right)^{10}$, 6th term.
10. $\left(c^2d^2 - \frac{x}{3}\right)^{15}$, 3d term.
11. $\left(4a^2 - \frac{b^2}{4}\right)^{13}$, 7th term.
12. $(m - n^2)^{14}$, 8th term.

14. **Exponents.** In the preceding discussions we have used only positive integers for exponents which, as shown in elementary algebra, obey the following laws.

$$a^m a^n = a^{m+n}. \qquad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$$

$$(a^m)^n = a^{mn}. \qquad \frac{a^m}{a^n} = a^{m-n}.$$

$$(ab)^m = a^m b^m.$$

Let us now assume that these formulas are true for all rational values of m and n, either positive, negative, fractional, or zero. We must then determine what meaning must be assigned in consequence to fractional, negative, and zero exponents. We know that a^6 means $a \times a \times a \cdots$ to 6 factors, but the meaning of $a^{2/3}$ is not so evident.

Let us consider $a^{p/q}$, where p and q are positive integers. From the above formulas we have

$$a^{p/q} \times a^{p/q} \times a^{p/q} \times \cdots \text{ to } q \text{ factors}$$
$$= a^{p/q+p/q+p/q+\cdots \text{ to } q \text{ terms}} = a^{(p/q)\times q} = a^p.$$

We have now shown that if we multiply the expression

$a^{p/q}$ by itself q times the result will be a^p. We must then, in accordance with the rules of elementary algebra, define $a^{p/q}$ as being the qth root of a^p. That is,

$$a^{p/q} = \sqrt[q]{a^p}.$$

Thus

$$a^{2/3} = \sqrt[3]{a^2}, \qquad a^{1/2} = \sqrt{a}, \qquad a^{3/4} = \sqrt[4]{a^3}, \quad \text{etc.}$$

Consider now a^0, where $a \neq 0$. We have

$$a^m \times a^0 = a^{m+0} = a^m.$$

Dividing by a^m, we find

$$a^0 = 1.$$

We must, therefore, define a^0 as being equal to 1, when a is not equal to zero. The special case where $a = 0$ will not be considered here.

The definition of a^{-m}, where m is a positive integer, or fraction, and $a \neq 0$, may be found as follows. We have

$$a^m \times a^{-m} = a^{m-m} = a^0 = 1.$$

Dividing by a^m gives

$$a^{-m} = \frac{1}{a^m}.$$

EXAMPLE 1. Multiply $2 + 3x^{1/3} - x^{2/3}$ by $3x^{-1/3} + 4$.

$$\begin{array}{l} 2 + 3x^{1/3} - x^{2/3} \\ 3x^{-1/3} + 4 \\ \hline 6x^{-1/3} + 9 - 3x^{1/3} \\ \qquad + 8 + 12x^{1/3} - 4x^{2/3} \\ \hline 6x^{-1/3} + 17 + 9x^{1/3} - 4x^{2/3} \end{array}$$

EXAMPLE 2. Solve the equation $x^{-3/2} = 8$.

We have

$$\frac{1}{x^{3/2}} = 8.$$

Hence

$$x^{3/2} = \frac{1}{8}.$$

Thus

$$\sqrt{x^3} = \frac{1}{8}, \quad \text{and} \quad x^3 = \left(\frac{1}{8}\right)^2 = \frac{1}{64}.$$

Then

$$x = \sqrt[3]{\frac{1}{64}} = \frac{1}{4}.$$

EXERCISES

Express with radical signs:

1. $a^{2/3}$.
2. $a^{4/5}$.
3. $b^{5/6}$.
4. $x^{3/4}$.
5. $(xy)^{1/2}$.
6. $(x^2y)^{1/3}$.
7. $(abc)^{2/3}$.
8. $(a^2b^2c)^{1/5}$.

Express with fractional exponents:

9. $\sqrt{a^3b^5}$.
10. $(\sqrt[5]{a^3})^{3/4}$.
11. $(\sqrt[9]{a^4b^8})^{3/8}$.
12. $(\sqrt[8]{a^{-3}b^4})^2$.
13. $\sqrt{2xy^{-5}}$.
14. $\sqrt[3]{4a^2b^6}$.
15. $\sqrt[3]{(a^2b^4)^4}$.
16. $\sqrt[5]{(x^{-3}y^{-4})^2}$.

Express with positive exponents:

17. $\dfrac{4x^{-2/3}y^{-1/2}}{5a^{-2}y^{-5}z^{-3}}$.
18. $\dfrac{6a^{-1/3}b^2c^{4/3}}{7a^{5/3}b^{-1/2}c^{-2/3}}$.
19. $\dfrac{x^8y^{-6}}{3a^{-2}x^{-3}}$.
20. $\dfrac{4a^{-3}b^{-1}c^{-2}}{5a^{-1}b^{-2}c^{-3}}$.
21. $\dfrac{3a^{-2}n^{-3}}{8b^{-4}}$.
22. $\dfrac{5x^{-3}y^4}{8y^{-2}z^{-3}}$.
23. $\dfrac{4x^{1/3}y^{-2/3}}{7x^{-1/4}y^{-1/3}}$.
24. $\dfrac{2^{-1}a^2n^{-3}}{3^{-2}b^{-2}n^2}$.
25. $\dfrac{2^{-3}a^{-2/3}b^{1/5}}{3^{-3}a^{1/3}b^{-4/5}}$.

Find the value of:

26. $8^{5/3}$.
27. $9^{3/2}$.
28. $\left(\dfrac{81}{16}\right)^{-5/4}$.
29. $4^{-3/2}$.
30. $16^{-5/4}$.
31. $\left(-\dfrac{1}{64}\right)^{2/3}$.
32. $(0.25)^{-1/2}$.
33. $8^{1/3} \cdot 4^{-1/2}$.
34. $\left(-\dfrac{1}{216}\right)^{-2/3}$.
35. 4^0.
36. $(-32)^{-3/5}$.
37. $\left(\dfrac{1}{64}\right)^{7/6}$.

Multiply:

38. $a^{7/4} - a^{3/2} + a^{5/4} - a$ by $a^{3/4} + a^{1/2}$.

39. $x^{3/2} + xy^{3/4} + x^{1/2}y^{3/2} + y^{9/4}$ by $x^{1/2} - y^{3/4}$.

40. $2x - 3 + x^{-1}$ by $3x + 2 - x^{-1}$.

41. $x^{3/2} - 4x^{3/4} - 5 + 6x^{-3/4}$ by $2x^{-3/4} + x^{-3/2} - 3x^{-9/4}$.

42. $3x^{3/5} - 4x^{1/5} - x^{-1/5}$ by $3x^{1/5} + x^{-1/5} - 6x^{-3/5}$.

43. $a^{4/5} - a^{3/5}b^{1/5} + a^{2/5}b^{2/5} - a^{1/5}b^{3/5} + b^{4/5}$ by $a^{1/5} + b^{1/5}$.

44. $x^{3/4} + x^{1/2}y^{1/4} + x^{1/4}y^{1/2} + y^{3/4}$ by $x^{1/4} - y^{1/4}$.

Divide:

45. $x - 1$ by $x^{1/3} - 1$.

46. $x^2 - 1$ by $x^{1/2} + 1$.

47. $a^{4/3} - 2 + a^{-4/3}$ by $a^{2/3} - a^{-2/3}$.

48. $x^{-1} + 4x^{1/4} + 6x^{3/2} + 4x^{11/4} + x^4$ by $x^{-1/4} + x$.

49. $15a - 13a^{1/3} - 10a^{-1/3} + 8a^{-1}$ by $5a^{2/3} + 4$.

50. $x^2y^{-4/3} - 10xy^{-2/3} + 9$ by $x^{3/2}y^{1/3} + 2xy^{2/3} - 3x^{1/2}y$.

51. $a^2 - 3a^{5/3} + 6a^{4/3} - 7a + 6a^{2/3} - 3a^{1/3} + 1$ by $a^{2/3} - a^{1/3} + 1$.

52. $4a^{5/4}b^{-2} - 17a^{1/4}b^2 + 16a^{-3/4}b^6$ by $2a^{1/2} - b^2 - 4a^{-1/2}b^4$.

Find the value of:

53. $(b^nx^{-m} - a^nx^m)^2$.

54. $(4a - 5a^{-1})(4a + 3a^{-1})$.

55. $(a + a^{1/2} - 4)(a + a^{1/2} + 4)$.

56. $(2a^{4/3} - 3b^{-1/2})^3$.

57. $(x^{1/2} - y^{-2/3})^6$.

58. $(a^{-1/4} + b^{1/4})^4$.

59. Find the fifth term of $\left(x^{-2/3}y^{5/6} + \frac{a}{2}\right)^8$.

60. Find the fourth term of $\left(\frac{a^{-3}}{4} - \frac{8}{x^{-3}}\right)^{10}$.

Simplify:

61. $(3a^{1/2}x^{-1/4}y^{1/6})^4$.

62. $(64a^{-3/2}x^{-3/4})^{4/3}$.

63. $\sqrt[3]{\frac{a^6b^{-3}y^6}{x^{-3}c^6z^9}}$.

64. $\sqrt[3]{a^2\sqrt{a^6}}$.

65. $\left(\frac{32a^{-1/2}}{x^3}\right)^{-2/5}$.

66. $\left(\frac{a^{-1}b}{a^{1/2}b^{-2/3}}\right)^{-2} \div \left(\frac{a^{-3}}{b^{-1}}\right)^{-1/3}$.

67. $\left(\frac{x^{-2}y^3}{x^3y^{-2}}\right)^{-1/5} \times \left(\frac{y^3x^{-3}}{x^3y^{-3}}\right)^{-1}$.

Solve the following equations:

68. $x^{2/3} = 25$.

69. $x^{3/5} = \frac{1}{64}$.

70. $x^{-2/3} = 16$.

15. **Radicals.** By making use of the results of § 14 we may derive the following formulas which are useful in simplifying many radical expressions. We have

$$a^{1/n} = \sqrt[n]{a}, \qquad b^{1/n} = \sqrt[n]{b}.$$

Multiplying these equations gives

$$a^{1/n} \cdot b^{1/n} = \sqrt[n]{a} \cdot \sqrt[n]{b}.$$

But

$$a^{1/n} \cdot b^{1/n} = (ab)^{1/n} = \sqrt[n]{ab}.$$

Thus

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}.$$

For example,

$$\sqrt{4 \times 9} = \sqrt{4} \cdot \sqrt{9},$$

that is,

$$\sqrt{36} = 2 \times 3 = 6.$$

Also

$$\left(\frac{a}{b}\right)^{1/n} = \frac{a^{1/n}}{b^{1/n}}.$$

Then

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

EXAMPLE 1. Simplify $\sqrt[3]{250}$.

$$\sqrt[3]{250} = \sqrt[3]{125 \times 2}$$
$$= \sqrt[3]{125} \cdot \sqrt[3]{2} = 5\sqrt[3]{2}.$$

EXAMPLE 2. Simplify $\sqrt{12a^2b^3}$.

$$\sqrt{12a^2b^3} = \sqrt{4a^2b^2} \cdot \sqrt{3b}$$
$$= 2ab\sqrt{3b}.$$

EXAMPLE 3. Simplify $\sqrt[3]{\frac{16a^4}{27b^3}}$.

$$\sqrt[3]{\frac{16a^4}{27b^3}} = \sqrt[3]{\frac{8a^3}{27b^3}} \cdot \sqrt[3]{2a}$$
$$= \frac{2a}{3b}\sqrt[3]{2a}.$$

EXAMPLE 4. Simplify $\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{6}}$.

Multiplying both numerator and denominator by $\sqrt{6}$ gives

$$\frac{\sqrt{2}\cdot\sqrt{6}+\sqrt{3}\cdot\sqrt{6}}{\sqrt{6}\cdot\sqrt{6}}=\frac{\sqrt{12}+\sqrt{18}}{\sqrt{36}}$$

$$=\frac{\sqrt{4}\cdot\sqrt{3}+\sqrt{9}\cdot\sqrt{2}}{6}=\frac{2\sqrt{3}+3\sqrt{2}}{6}.$$

EXAMPLE 5. Simplify $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$.

Multiplying both numerator and denominator by $2+\sqrt{3}$ gives

$$\frac{(2+\sqrt{3})(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}=\frac{4+4\sqrt{3}+3}{4-3}=\frac{7+4\sqrt{3}}{1}=7+4\sqrt{3}.$$

EXERCISES

Reduce each of the following expressions to its simplest form:

1. $\sqrt[4]{144}$.
2. $\sqrt[3]{54}$.
3. $2\sqrt{72}$.
4. $\sqrt[4]{162}$.
5. $\sqrt[6]{25}$.
6. $\sqrt[5]{243a^6b^7}$.
7. $2\sqrt{(a+b)^3}$.
8. $4\sqrt[3]{(x-y)^5}$.
9. $\sqrt[3]{5a^5b^6}$.
10. $\sqrt{500x^7a^3b}$.
11. $\sqrt{\dfrac{1}{40}}$.
12. $\sqrt{\dfrac{9a^2b}{16x^2y^6}}$.
13. $\sqrt{\dfrac{16a^8}{45x^3b^5}}$.
14. $\sqrt[6]{\dfrac{128x^8y^6}{81a^4b^5}}$.

Simplify:

15. $\sqrt{32}+\sqrt{72}-\sqrt{18}$.
16. $2\sqrt{8}+5\sqrt{72}-7\sqrt{18}$.
17. $\sqrt[3]{192a}-\sqrt[3]{3a}$.
18. $\sqrt{(a^2-1)(a+1)}+\sqrt{4a-4}$.
19. $\sqrt{\dfrac{25}{18}}+\sqrt{\dfrac{40}{27}}+\sqrt{\dfrac{5}{6}}$.
20. $4\sqrt[3]{54}+9\sqrt[3]{250}-2\sqrt[3]{686}$.
21. $\sqrt[3]{16}+\sqrt[3]{54}+\sqrt[3]{2662}-\sqrt[3]{128}+\sqrt[3]{-250}$.
22. $4\sqrt{45}+2\sqrt{48}-5\sqrt{27}+4\sqrt{20}-2\sqrt{12}$.

23. $\sqrt{a^3 - a^2b} - \sqrt{ab^2 - b^3} + \sqrt{(a+b)(a^2 - b^2)}$.

Multiply:

24. $\sqrt{12}$ by $\sqrt{6}$.

25. $3\sqrt{45}$ by $4\sqrt{72}$.

26. $2\sqrt{32}$ by $3\sqrt{8}$.

27. $\sqrt{8} + \sqrt{5}$ by $\sqrt{8} - \sqrt{5}$.

28. $\sqrt{3} - 2\sqrt{6}$ by $2\sqrt{3} + 3\sqrt{6}$.

29. $\sqrt{5} + 3\sqrt{2}$ by $2\sqrt{5} - 4\sqrt{2}$.

30. $\sqrt{2} + \sqrt{3} - \sqrt{5}$ by $\sqrt{2} + \sqrt{3} + \sqrt{5}$.

31. $\sqrt{3} - 3\sqrt{2} + 2\sqrt{5}$ by $2\sqrt{3} + \sqrt{2} - 3\sqrt{5}$.

Divide:

32. $\sqrt{120}$ by $\sqrt{6}$.

33. $\sqrt{8} + 5\sqrt{12}$ by $\sqrt{2}$.

34. $8\sqrt{12} - 12\sqrt{3}$ by $2\sqrt{3}$.

35. $10\sqrt{14} - 5\sqrt{63} + 4\sqrt{28} + 6\sqrt{7}$ by $2\sqrt{7}$.

Expand:

36. $(2\sqrt{5} - \sqrt{6})^2$.

37. $(4\sqrt{3} - \sqrt{5})^3$.

38. $(\sqrt{3} - 2)^4$.

39. $(2\sqrt{5} - 3\sqrt{2})^4$.

40. $(a + \sqrt{bc})^5$.

41. $(\sqrt{a+1} + 2)^3$.

Simplify:

42. $\dfrac{4 - \sqrt{3}}{4 + \sqrt{3}}$.

43. $\dfrac{1 + \sqrt{6}}{1 - \sqrt{6}}$.

44. $\dfrac{2 - \sqrt{2}}{3 + \sqrt{2}}$.

45. $\dfrac{a + \sqrt{b}}{a - \sqrt{b}}$.

46. $\dfrac{\sqrt{a+1} - 2}{\sqrt{a+1} + 2}$.

47. $\dfrac{\sqrt{x+y} - \sqrt{x-y}}{\sqrt{x+y} + \sqrt{x-y}}$.

Taking $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, $\sqrt{5} = 2.236$, and $\sqrt{6} = 2.449$, find the value of each of the following expressions:

48. $\dfrac{1}{\sqrt{5} - \sqrt{3}}$.

49. $\dfrac{1}{\sqrt{3} + \sqrt{2}}$.

50. $\dfrac{5}{3 - \sqrt{3}}$.

51. $\dfrac{\sqrt{2} + \sqrt{3}}{2\sqrt{2} + \sqrt{3}}$.

16. Complex Numbers. In the preceding discussions we have in every case assumed the existence of the number $\sqrt[n]{a}$, where n is any positive integer and where a is either positive, negative, fractional, or zero. That is, we have assumed that the number represented by $\sqrt[n]{a}$ is one of the rational, or irrational, numbers which constitute our number-system (§ 1), and that its exact value, if irrational, may be approximated to any degree of accuracy. For example, $\sqrt{8} = 2.82843$ to five places of decimals.

If n is an even number and a is negative, the value of $\sqrt[n]{a}$ cannot even be approximated. For example, if we denote $\sqrt{-4}$ by x, we must find a number x whose square is -4. It is evident that if x be either positive or negative its square will be positive. There is, therefore, no number in our number-system which will represent $\sqrt{-4}$. We must then either enlarge our number-system or say that the operation of extracting an even root of a number cannot be performed on a negative number.

There are many advantages in having a number-system in which it is possible to apply any of the operations of addition, subtraction, multiplication, division, raising to powers, and extracting roots, to rational or irrational numbers. We shall, therefore, expand our number-system so as to include all the numbers which result from these operations.

These new numbers, such as $\sqrt{-4}$, $\sqrt[4]{-7}$, $\sqrt[6]{-2}$, are called ***imaginary numbers*** or ***pure imaginary numbers.*** All other numbers, such as 2, -7, $3/2$, $-6/5$, $\sqrt{12}$, $\sqrt[3]{-10}$, may now be called ***real numbers.*** The algebraic sum of a real and a pure imaginary number, such as $3 + \sqrt{-4}$, is called a ***complex number.*** It is evident that

$$\sqrt{-4} = \sqrt{4} \cdot \sqrt{-1} = 2\sqrt{-1},$$
$$3 + \sqrt{-6} = 3 + \sqrt{6} \cdot \sqrt{-1},$$
$$2 - \sqrt{-12} = 2 - 2\sqrt{3} \cdot \sqrt{-1}, \text{ etc.}$$

In fact, it can be shown that any complex number may be written in the form

$$a + b\sqrt{-1},$$

where a and b are real (positive or negative) numbers, although the proof is beyond the scope of this book.

It is also possible to prove that the result of operating on numbers of the form $a + b\sqrt{-1}$ with any of the six operations mentioned above is a number of the same form. Since these are the only operations we shall need in this book, no further extension of our number-system will be necessary.

For brevity we shall write i for $\sqrt{-1}$. Any algebraic number may, therefore, be written in the form

$$a + bi,$$

where a and b are real. This complex number reduces to a pure imaginary number if $a = 0$, and to a real number if $b = 0$.

If the pure imaginary number $i(=\sqrt{-1})$ be raised to successive powers, we find

$$\begin{aligned} i &= \sqrt{-1}, \\ i^2 &= (\sqrt{-1})^2 = -1, \\ i^3 &= i^2 \times i = (-1)i = -i, \\ i^4 &= (i^2)^2 = (-1)^2 = +1, \\ i^5 &= i^4 \times i = (+1)i = i, \text{ etc.} \end{aligned}$$

That is, any positive integral power of i is either equal to $+1$, -1, i, or $-i$. For example,

$$i^{18} = i^{16} \times i^2 = (i^4)^4 \times i^2 = (+1)^4(-1) = -1.$$

Example 1. Multiply $2 + \sqrt{-1}$ by $3 - \sqrt{-1}$.

$$(2 + i)(3 - i) = 6 + 3i - 2i - i^2 = 6 + i - (-1) = 7 + i.$$

EXAMPLE 2. Multiply $\sqrt{-3}$ by $\sqrt{-4}$.

$$\sqrt{-3} = i\sqrt{3}, \qquad \sqrt{-4} = i\sqrt{4} = 2i.$$

Hence

$$\sqrt{-3} \times \sqrt{-4} = (i\sqrt{3})(2i) = 2i^2\sqrt{3} = -2\sqrt{3}.$$

NOTE. If we make use of the formula $\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$ (§ 15), we obtain

$$\sqrt{-3} \times \sqrt{-4} = \sqrt{(-3)(-4)} = \sqrt{+12} = +2\sqrt{3}.$$

Where is the error?

EXAMPLE 3. Simplify $\dfrac{2+i}{1-i}$.

Multiplying the numerator and denominator by $1 + i$ gives

$$\frac{(2+i)(1+i)}{(1-i)(1+i)} = \frac{2+3i+i^2}{1-i^2} = \frac{2+3i-1}{1-(-1)} = \frac{1+3i}{2}.$$

EXERCISES

Simplify:

1. $(\sqrt{-1})^{14}$.
2. $(\sqrt{-1})^{23}$.
3. $(\sqrt{-1})^{13}$.
4. $2\sqrt{-25} + 4\sqrt{-9} - 3\sqrt{-25}$.
5. $3\sqrt{-8} - 5\sqrt{-32} + 4\sqrt{-128}$.
6. $4\sqrt{-81} - 2\sqrt{-144} + 3\sqrt{-256}$.

Multiply:

7. $\sqrt{-15}$ by $\sqrt{20}$.
8. $\sqrt{-3}$ by $\sqrt{-12}$.
9. $\sqrt{-6}$ by $\sqrt{-54}$.
10. $\sqrt{12} - \sqrt{-9}$ by $\sqrt{3} + \sqrt{-25}$.
11. $\sqrt{5} + 2\sqrt{-6}$ by $\sqrt{5} - 2\sqrt{-6}$.
12. $\sqrt{-4} - 3\sqrt{-9}$ by $\sqrt{-4} + 4\sqrt{-25}$.

Divide:

13. $\sqrt{-12}$ by $\sqrt{6}$.
14. $\sqrt{27}$ by $\sqrt{-3}$.
15. $\sqrt{-20}$ by $\sqrt{-5}$.
16. $\sqrt{-441}$ by $\sqrt{-3}$.

Simplify:

17. $(\sqrt{-9} + 3\sqrt{-4})^3$.
18. $\dfrac{3+2i}{2+i}$.
19. $(2-i)^4$.
20. $\dfrac{1}{i+i^5}$.
21. $(\sqrt{3} - \sqrt{-2})^3$.
22. $\dfrac{a+bi}{a-bi}$.

CHAPTER II

THE QUADRATIC EQUATION

17. Definitions. An equation which contains the square of the unknown quantities, but no higher power, is called a ***quadratic equation*** or an ***equation of the second degree.*** Any quadratic equation, involving only one unknown number x, may be reduced to the form

$$ax^2 + bx + c = 0,$$

where a, b, c represent known numbers and $a \neq 0$.

If the equation contains both the square and the first power of the unknown, it is called a *complete*, or an *affected, quadratic equation.* If it contains only the square of the unknown, it is called an *incomplete*, or *pure, quadratic equation.*

18. Solution by Factoring. When the left member of the equation can be readily factored, the solution is easily obtained. Consider the equation

$$x^2 + 2x - 15 = 0,$$

which may be written in the form

$$(x - 3)(x + 5) = 0.$$

This equation merely states in algebraic language that the product of the two numbers $x - 3$ and $x + 5$ is equal to zero. This can be true only when one, or both, of these numbers is equal to zero. Thus we are sure that either

$$x - 3 = 0, \quad \text{that is,} \quad x = 3;$$

or else

$$x + 5 = 0, \quad \text{that is,} \quad x = -5.$$

It is evident that there is no one value of x which will make both $x - 3$ and $x + 5$ equal to zero. In order to check these results we must substitute each value of x in the original equation and show that the result is an identity. We obtain

$$9 + 6 - 15 = 0, \qquad \text{when} \qquad x = 3,$$

and

$$25 - 10 - 15 = 0, \qquad \text{when} \qquad x = -5.$$

The numbers 3 and -5 are, therefore, the roots of the given equation.

This same method may be used for solving an equation of higher degree than the second. In such problems the method of synthetic division is very useful.

EXAMPLE. Solve the equation $18x^2 - 23x - 6 = 0$.

This equation may be written in the form

$$(2x - 3)(9x + 2) = 0.$$

Whence

$$2x - 3 = 0, \qquad \text{and} \qquad x = 3/2,$$

or

$$9x + 2 = 0, \qquad \text{and} \qquad x = -2/9.$$

CHECK.

$$18\left(\frac{9}{4}\right) - 23\left(\frac{3}{2}\right) - 6 = 0, \qquad \text{or} \qquad \frac{81}{2} - \frac{69}{2} - 6 = 0,$$

or

$$\frac{12}{2} - 6 = 0,$$

$$18\left(\frac{4}{81}\right) - 23\left(-\frac{2}{9}\right) - 6 = 0, \qquad \text{or} \qquad \frac{8}{9} + \frac{46}{9} - 6 = 0,$$

or

$$\frac{54}{9} - 6 = 0.$$

EXERCISES

Find all the roots of each of the following equations.

1. $a^2 - 4a = 5$.
2. $x^2 - a^2 = 0$.
3. $t^2 - 15t = 0$.
4. $6x^2 + 7x + 1 = 0$.
5. $2t^2 - t - 36 = 0$.
6. $5s - 6s^2 + 14 = 0$.
7. $6a^2 + 5a - 6 = 0$.
8. $3x^3 - 2x - 8 = 0$.
9. $7a^2 + 20a - 32 = 0$.
10. $2x^2 - 5ax + 2a^2 = 0$.
11. $\frac{x^2}{10} - \frac{2x}{5} - \frac{9}{2} = 0$.
12. $\frac{x(9 + 2x)}{15} = \frac{3x + 6}{5}$.
13. $x^3 = 9x$.
14. $x^3 = a^2x$.
15. $(a + 1)(a - 1) = 195$.
16. $(x + a)^3 - (x - a)^3 = 8a^3$.
17. $x^3 + x^2 - 4x - 4 = 0$.
18. $a(a^2 - 4) - 3(a - 2) = 0$.
19. $x^4 - 81x^2 = 0$.
20. $x^4 - 17x^2 + 16 = 0$.
21. $x^3 - 6x^2 + 11x - 6 = 0$.
22. $x + \frac{1}{a} = a + \frac{1}{x}$.
23. $\frac{3x^2 + 7}{7} - \frac{5x^2 + 3}{14} - \frac{4x^2 - 10}{35} = 0$.
24. $\frac{x + 5a + b}{x - 3a + b} + \frac{x - a + 4b}{x - a + 2b} = 0$.

19. **Solution by Completing the Square.** If the left member of the equation cannot be readily factored, we use a method which is known as *completing the square*. In applying this method we first reduce the given equation to such a form that the left member is a perfect square, and then take the square root of both members. We shall show two ways of completing the square in any equation such as

$$3x^2 - 17x + 10 = 0,$$

which may be written in the form

$$3x^2 - 17x = -10.$$

First Solution. Dividing both members of the equation by 3, we find

$$x^2 - \frac{17}{3}x = -\frac{10}{3}.$$

Adding to both members the square of one-half the coefficient of x, we have

$$x^2 - \frac{17}{3}x + \left(\frac{17}{6}\right)^2 = -\frac{10}{3} + \frac{289}{36} = \frac{-120 + 289}{36} = \frac{169}{36}.$$

Extracting the square root of both members of the equation, we get

$$x - \frac{17}{6} = \pm\frac{13}{6},$$

where the double sign indicates that the square root of 169/36 may be either + 13/6 or − 13/6. Hence either

$$x - \frac{17}{6} = \frac{13}{6}, \quad \text{and} \quad x = \frac{17 + 13}{6} = \frac{30}{6} = 5,$$

or

$$x - \frac{17}{6} = -\frac{13}{6}, \quad \text{and} \quad x = \frac{17 - 13}{6} = \frac{4}{6} = \frac{2}{3}.$$

Hence the roots of the given equation are 5 and 2/3.

We may now state the following rule.

Reduce the equation to the form $x^2 + px = q$.

Complete the square by adding to both members the square of one-half the coefficient of x.

Equate the positive square root of the first member to each of the square roots of the second, and solve the resulting linear equations.

Second Solution. Multiplying the given equation by four times the coefficient of x^2, that is, by 12, we obtain

$$36x^2 - 204x = -120.$$

Adding to both members the square of the coefficient of x in the given equation, we find

$$36x^2 - 204x + (17)^2 = -120 + 289 = 169.$$

Extracting the square root of both members, we have

$$6x - 17 = \pm 13.$$

Whence

$$6x - 17 = 13, \qquad 6x = 30, \qquad x = 5,$$

or

$$6x - 17 = -13, \qquad 6x = 4, \qquad x = 2/3.$$

This method of course gives the same results as the first, but it avoids the use of fractions in completing the square.

This rule may now be stated as follows.

Reduce the equation to the form $ax^2 + bx = -c$.

Multiply both members by four times the coefficient of x^2, *and add to each the square of the coefficient of* x *in the given equation.*

Extract the square root of both members and proceed as before.

When the equation is transformed by either of these methods, the left member of the resulting equation will always be a perfect square. This is not always true however of the right member, as will be seen in the following examples.

EXAMPLE 1. Solve the equation $x^2 - 3x + 1 = 0$. Transposing the third term and multiplying by 4 gives

$$4x^2 - 12x = -4.$$

Add 3^2 $(= 9)$ to both members.

$$4x^2 - 12x + 9 = -4 + 9 = 5.$$

Extract the square root.

$$2x - 3 = \pm \sqrt{5}.$$

Whence

$$2x - 3 = \sqrt{5}, \qquad 2x = 3 + \sqrt{5}, \qquad x = \frac{3 + \sqrt{5}}{2},$$

or

$$2x - 3 = -\sqrt{5}, \qquad 2x = 3 - \sqrt{5}, \qquad x = \frac{3 - \sqrt{5}}{2}.$$

The roots are therefore irrational numbers, whose approximate values may be obtained from the tables.

Thus

$$x = \frac{3 + 2.23607}{2} = \frac{5.23607}{2} = 2.61803,$$

and

$$x = \frac{3 - 2.23607}{2} = \frac{0.76393}{2} = 0.38196$$

CHECK. Substituting $(3 + \sqrt{5})/2$ for x in the given equation, we find

$$\left(\frac{3 + \sqrt{5}}{2}\right)^2 - 3\left(\frac{3 + \sqrt{5}}{2}\right) + 1 = \frac{9 + 6\sqrt{5} + 5}{4} - \frac{9 + 3\sqrt{5}}{2} + 1$$

$$= \frac{14 + 6\sqrt{5}}{4} - \frac{9 + 3\sqrt{5}}{2} + 1 = \frac{7 + 3\sqrt{5}}{2} - \frac{9 + 3\sqrt{5}}{2} + 1$$

$$= \frac{7 + 3\sqrt{5} - 9 - 3\sqrt{5}}{2} + 1 = \frac{-2}{2} + 1 = 0.$$

The substitution of $(3 - \sqrt{5})/2$ for x likewise reduces the left member of the equation to zero.

EXAMPLE 2. Solve the equation $x^2 + x + 1 = 0$. Transposing the third term and multiplying by 4 gives

$$4x^2 + 4x = -4.$$

Add 1^2 $(= 1)$ to both members.

$$4x^2 + 4x + 1 = -3.$$

Extract the square root.

$$2x + 1 = \pm\sqrt{-3}.$$

Whence

$$2x = -1 + \sqrt{-3}, \qquad x = \frac{-1 + \sqrt{-3}}{2},$$

or

$$2x = -1 - \sqrt{-3}, \qquad x = \frac{-1 - \sqrt{-3}}{2}.$$

The roots are, therefore, complex numbers which may be written in the form

$$x = \tfrac{1}{2}(-1 + i\sqrt{3}), \qquad x = \tfrac{1}{2}(-1 - i\sqrt{3}).$$

CHECK. Substituting $\frac{-1+i\sqrt{3}}{2}$ for x in the given equation, we find

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^2+\frac{-1+i\sqrt{3}}{2}+1$$

$$=\frac{1-2i\sqrt{3}-3}{4}+\frac{-1+i\sqrt{3}}{2}+1$$

$$=\frac{-2-2i\sqrt{3}}{4}+\frac{-1+i\sqrt{3}}{2}+1$$

$$=\frac{-1-i\sqrt{3}}{2}+\frac{-1+i\sqrt{3}}{2}+1=\frac{-2}{2}+1=0.$$

The student should now substitute the other value for x and show that the left member reduces to zero.

EXERCISES

Solve:

1. $x^2-3x-54=0.$
2. $y^2+26y+160=0.$
3. $y^2+5y-25=0.$
4. $3a^2-2a-280=0.$
5. $3x^2-7x-16=0.$
6. $7a^2+9a-10=0.$
7. $m^2-15m-76=0.$
8. $2a^2+3a-14=0.$
9. $y^2+4y-50=0.$
10. $5a^2-2a+30=0.$
11. $2x^2+5x-33=0.$
12. $\frac{5a-2}{2a+1}=\frac{3a+5}{a-1}.$
13. $\frac{a+2}{a-1}+\frac{a-4}{2a}=\frac{7}{3}.$
14. $\frac{1+x}{2+x}-\frac{1}{6}=\frac{3-x}{5-x}.$
15. $\frac{x+a}{x-2a}+\frac{x-2a}{x+a}=1.$
16. $\frac{5}{a-2}-\frac{4}{a}=\frac{3}{a+6}.$
17. $28a-32a^2-3=0.$
18. $2a+4=3a^2-2.$
19. $5y^2-9y-18=0.$
20. $x^2-4abx=(a^2-b^2)^2.$
21. $x^2+m^2x^2+8mx+55=0.$
22. $x^2-2x\sqrt{ab}=15ab.$
23. $x^2-2ax+a^2-b^2=0.$
24. $x^2-2ax+6a-9=0.$
25. $a^2y^2-2ay-8=0.$
26. $(a+1)^3-a^3=1.$

27. $(2x + 1)^3 - (2x + 3)^3 + 386 = 0.$

28. $\dfrac{x - a}{x - b} - \dfrac{x - b}{x - a} + \dfrac{4ab}{a^2 - b^2} = 0.$

29. $\dfrac{2a - 1}{a + 1} + \dfrac{3a - 1}{a + 2} - \dfrac{a - 7}{a - 1} = 4.$ **30.** $\dfrac{x + 1}{c} - \dfrac{2}{cx} = \dfrac{x + 2}{ax - bx}.$

31. $\dfrac{x - 2}{x - 3} + \dfrac{3x - 11}{x - 4} = \dfrac{4x + 13}{x + 1}.$ **32.** $\dfrac{2x}{x - 1} + \dfrac{3x - 1}{x + 2} = \dfrac{5x - 11}{x - 2}.$

20. **Solution by Formula.** We have seen that the methods of § 19 apply to the quadratic equation with general (literal) coefficients as well as to one with numerical coefficients. The most general form of such an equation is

$$ax^2 + bx + c = 0.$$

Transpose the term which is independent of x and multiply by $4a$.

$$4a^2x^2 + 4abx = -4ac.$$

Adding to both members the square of b, we find

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac,$$

whence

$$2ax + b = \pm \sqrt{b^2 - 4ac},$$

and

$$2ax = -b \pm \sqrt{b^2 - 4ac}.$$

The two roots of the equation are, therefore,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This result may be used as a formula for the solution of any quadratic equation of the form $ax^2 + bx + c = 0$, where a, b, c may have any value ($a \neq 0$).

EXAMPLE 1. Solve the equation $6x^2 + 7x - 3 = 0$.

Here $a = 6$, $b = 7$, $c = -3$. Substituting these values in the formula gives

$$x = \frac{-7 \pm \sqrt{49 - (-72)}}{12} = \frac{-7 \pm \sqrt{121}}{12} = \frac{-7 \pm 11}{12}.$$

Thus the two solutions are

$$x = \frac{-7 + 11}{12} = \frac{4}{12} = \frac{1}{3}, \quad \text{and} \quad x = \frac{-7 - 11}{12} = \frac{-18}{12} = \frac{-3}{2}.$$

EXAMPLE 2. Solve the equation $2x^2 - 3x + 7 = 0$.

Here $a = 2$, $b = -3$, $c = 7$. Then

$$x = \frac{3 \pm \sqrt{9 - 56}}{4} = \frac{3 \pm \sqrt{-47}}{4}.$$

Thus

$$x = \frac{3 + i\sqrt{47}}{4}, \quad \text{and} \quad x = \frac{3 - i\sqrt{47}}{4}.$$

EXAMPLE 3. Solve the equation $mx^2 - (m^2 - 1)x - m = 0$.

Here $a = m$, $b = -(m^2 - 1)$, $c = -m$. Then

$$x = \frac{(m^2 - 1) \pm \sqrt{(m^2 - 1)^2 + 4m^2}}{2m} = \frac{(m^2 - 1) \pm \sqrt{m^4 + 2m^2 + 1}}{2m}$$

$$= \frac{(m^2 - 1) \pm (m^2 + 1)}{2m}.$$

Thus

$$x = \frac{m^2 - 1 + m^2 + 1}{2m} = \frac{2m^2}{2m} = m,$$

and

$$x = \frac{m^2 - 1 - m^2 - 1}{2m} = \frac{-2}{2m} = -\frac{1}{m}.$$

EXERCISES

Solve each of the following equations by the formula.

1. $a^2 - 4a + 11 = 0$.
2. $m^2 - 8m + 35 = 0$.
3. $4t^2 - 7t + 5 = 0$.
4. $7y^2 + 9y - 10 = 0$.

5. $9x^2 + 3x - 2 = 0$.

6. $6y^2 + 11y - 10 = 0$.

7. $4t^2 - 3t + 2 = 0$.

8. $\dfrac{5x - 7}{7x - 5} = \dfrac{x - 5}{2x - 13}$.

9. $\dfrac{3a - 8}{a - 2} = \dfrac{5a - 2}{a + 5}$.

10. $\dfrac{x - a}{x + a} - \dfrac{x + a}{x - a} = \dfrac{x^2 - 5a^2}{x^2 - a^2}$.

11. $\dfrac{a}{a + 2} - \dfrac{a}{a + 3} = \dfrac{a^2 + 2a - 2}{a^2 + 5a + 6}$.

12. $\dfrac{2m + 1}{m + 1} - 1 = \dfrac{1}{(m + 1)^2}$.

13. $a^2x^2 + 4ax - 21 = 0$.

14. $x^2 - 2ax + a^2 + b^2 = 0$.

15. $ax^2 - 5x - 2ax + 10 = 0$.

16. $(a - b)x^2 - ax + b = 0$.

17. $x^2 - 2m(m + n) + (m - n)x = 0$.

18. $(x + 1)(2x + 3) = 4x^2 - 22$.

19. $(x + a)^3 - (x - b)^3 = (a + b)^3$.

20. $\dfrac{a}{2x} + \dfrac{2x}{a} = 2b$.

21. $\dfrac{1}{x - a} + \dfrac{1}{x - b} = \dfrac{1}{a} + \dfrac{1}{b}$.

22. $\dfrac{1}{a^2 - 3a} - \dfrac{1}{a^2 + 4a} = \dfrac{14}{15a^2}$.

23. $\dfrac{3x - 4}{x - 4} + \dfrac{5x - 7}{2x - 2} = \dfrac{9x^2 - 38}{2x^2 - 10x + 8}$.

24. $\dfrac{1}{a + b + x} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{x}$.

25. $2(a + 1)x^2 - (a + 1)^2x - 4x + 2(a + 1) = 0$.

26. $abx^2 - (a + b)(ab + 1)x + (ab + 1)^2 = 0$.

27. $(a + b + c)x^2 - (2a + b + c)x + a = 0$.

21. **Formation of Quadratic Equations Having Given Roots.** We have just derived a general method for the solution of any quadratic equation. Suppose now that we wish to determine the quadratic equation which has two given numbers as its roots. This is always possible by the method used in the following example:

EXAMPLE. Determine the quadratic equation whose roots are 3 and -6.

We have $x = 3$, or $x - 3 = 0$; and $x = -6$, or $x + 6 = 0$. Hence the required equation is

$$(x - 3)(x + 6) = 0,$$

or

$$x^2 + 3x - 18 = 0,$$

which is obviously satisfied when either $x = 3$ or $x = -6$.

In a similar manner it may be shown that the equation whose roots are p and q is

$$(x - p)(x - q) = 0, \quad \text{or} \quad x^2 - (p + q)x + pq = 0.$$

EXERCISES

Form the quadratic equations whose roots are:

1. $3, -2$.
2. $-1, -3$.
3. $0, 4$.
4. $-\frac{1}{2}, \frac{2}{3}$.
5. $\frac{3}{4}, -2$.
6. $\frac{m}{n}, \frac{n}{m}$.
7. $2 + \sqrt{3}, 2 - \sqrt{3}$.
8. $i, -i$.
9. $a + b, a - b$.
10. $a + \sqrt{b}, a - \sqrt{b}$.
11. $3 + i, 3 - i$.
12. $\frac{\sqrt{a} + 3\sqrt{b}}{2}, \frac{\sqrt{a} - 3\sqrt{b}}{2}$.
13. $\frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$.
14. $\frac{3 + \sqrt{-31}}{2}, \frac{3 - \sqrt{-31}}{2}$.

22. **The Number of Roots of a Quadratic Equation.** From the formula of § 20 it is evident that every quadratic equation has two different roots unless $b^2 - 4ac = 0$. However, even in this case, if the equation be solved by factoring, it may be said that it has two roots which are equal.

Thus in the equation

$$x^2 - 6x + 9 = 0,$$

$a = 1$, $b = -6$, $c = 9$, and $b^2 - 4ac = 36 - 36 = 0$. The general formula apparently then gives only one root $x = 3$. But if we write the equation in the form $(x - 3)(x - 3) = 0$, we have $x - 3 = 0$ or $x = 3$, and $x - 3 = 0$ or $x = 3$.

We shall now show that a quadratic equation cannot have more than two different roots. In other words it is not possible to find more than two different numbers which satisfy the equation.

Suppose r_1 and r_2 are two different roots of the equation

$$ax^2 + bx + c = 0.$$

Then $x - r_1$ and $x - r_2$ must be factors of the left member. That is,

$$ax^2 + bx + c = a(x - r_1)(x - r_2).$$

Let us now assume that the equation has a third root which we may denote by r_3. Then

$$a(r_3 - r_1)(r_3 - r_2) = 0.$$

Since $a \neq 0$, either $r_3 - r_1 = 0$ and $r_3 = r_1$, or $r_3 - r_2 = 0$ and $r_3 = r_2$. Hence, if each of three numbers, r_1, r_2, and r_3, satisfies a given quadratic equation, at least two of them must be equal.

From the foregoing results it is evident that every quadratic equation has two, and only two, roots, which may however be equal.

The equation in the example below apparently has only one root. The student should examine the solution carefully and find out what became of the other one.

EXAMPLE. Solve the equation

$$(x - 3)(x + 4) + 2(x + 2)(x + 3) = 0.$$

Performing the indicated multiplications gives

$$x^2 + x - 12 + 2x^2 + 10x + 12 = 0,$$

or

$$3x^2 + 11x = 0.$$

Divide by $3x$.

$$x + \frac{11}{3} = 0, \qquad \text{or} \qquad x = -\frac{11}{3}.$$

23. **The Character of the Roots.** It follows from § 20 that the roots of the equations $ax^2 + bx + c = 0$ and $ay^2 + by + c = 0$ are the same. That is, the roots of any quadratic equation depend only on the values of the coefficients a, b, and c, and are independent of the letter which is used to represent the unknown quantity.

We shall now discuss the character of the roots of the quadratic equation

$$ax^2 + bx + c = 0,$$

where a, b, and c are any real numbers ($a \neq 0$). The roots of this equation are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The expression $b^2 - 4ac$ is called the ***discriminant*** of the quadratic. It is obvious that the character of the roots depends upon the value of the discriminant and that, if a, b, and c are real numbers,

(1) if $b^2 - 4ac$ is *positive*, the roots are *real* and *unequal;*

(2) if $b^2 - 4ac$ is *zero*, the roots are *real* and *equal;*

(3) if $b^2 - 4ac$ is *negative*, the roots are *complex*.

If a, b, and c are rational numbers, we may also add that the roots are

(4) *rational*, if $b^2 - 4ac$ is a perfect square;

(5) *irrational*, if $b^2 - 4ac$ is not a perfect square.

Example. Determine by inspection the nature of the roots of the equation $6x^2 + x - 2 = 0$.

Here $a = 6, b = 1, c = -2$; and $b^2 - 4ac = 1 - (-48) = 49$.
Since $b^2 - 4ac$ is positive, the roots are real and unequal.
Since $b^2 - 4ac$ is a perfect square, both roots are rational.

24. **Relation between the Roots and the Coefficients.** It is easily seen that

$$r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a},$$

and

$$r_1 r_2 = \frac{(-b + \sqrt{b^2 - 4ac})(-b - \sqrt{b^2 - 4ac})}{4a^2}$$

$$= \frac{(-b)^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

Hence

(1) *The* ***sum*** *of the roots of any quadratic equation is equal to the coefficient of x with its sign changed, divided by the coefficient of x^2;*

(2) *The* ***product*** *of the roots of any quadratic equation is equal to the constant term divided by the coefficient of x^2.*

EXERCISES

Determine by inspection the sum, the product, and the nature of the roots of each of the following equations.

1. $18x^2 + 27x - 11 = 0$.
2. $8x^2 - 4x - 13 = 0$.
3. $25x^2 - 20x + 4 = 0$.
4. $s^2 + 3s + 5 = 0$.
5. $y^2 + 7 = 0$.
6. $9y^2 - 30y + 25 = 0$.
7. $4x^2 - 4x - 1 = 0$.
8. $3y^2 - 4y + 2 = 0$.
9. $20t^2 - 41t + 20 = 0$.
10. $10 + 12a - 15a^2 = 0$.
11. Show that the roots of the equation

$$(a - b + c)x^2 + 2cx - (a - b - c) = 0$$

are rational.

12. One root of $15y^2 + 28y - 32 = 0$ is 4/5. Find the other.
13. One root of $3x^2 + 7x - 6 = 0$ is 2/3. Find the other.

For what values of k will the roots of the following equations be equal?

14. $x^2 + kx + 16 = 0$.

15. $x^2 + 12x + 8k = 0$.

16. $kx^2 + kx + 1 = 0$.

17. $9x^2 + kx + x + 1 = 0$.

18. $2x^2 + (1 + k)x + 2 = 0$.

19. $(2x + k)^2 = 3(x - 5)$.

20. $(k - 2)x^2 + (k - 5)x - 5 = 0$.

21. $(3k + 1)y^2 + (2k + 2)y + k = 0$.

22. $a^2(mx + k)^2 + b^2x^2 = a^2b^2$.

23. $(3k + 1)x^2 + (11k + 1)x + 8k + 9 = 0$.

If r_1 and r_2 are the roots of the equation $ax^2 + bx + c = 0$, find the value of:

24. $\frac{1}{r_1} + \frac{1}{r_2}$.

25. $\frac{r_1}{r_2} + \frac{r_2}{r_1}$.

26. $r_1^2 + r_2^2$.

27. $(r_1 - r_2)^2$.

28. $r_1^2 + r_1r_2 + r_2^2$.

29. $\frac{1}{r_1^2} + \frac{1}{r_2^2}$.

If the roots of $ax^2 + bx + c = 0$ are r_1 and r_2, form the equations whose roots are:

30. $-r_1, -r_2$.

31. $2r_1, 2r_2$.

32. $3r_1, 3r_2$.

33. $\frac{1}{r_1}, \frac{1}{r_2}$.

34. r_1^2, r_2^2.

35. $\frac{1}{r_1^2}, \frac{1}{r_2^2}$.

36. Show that one root of $ax^2 + bx + c = 0$ will be n times the other if

$$b^2 = \frac{(n + 1)^2}{n} \cdot ac.$$

25. **Equations in Quadratic Form.** It is often possible to reduce an equation to the form of a quadratic by the use of a new letter. Such an equation is said to be in *quadratic form.*

EXAMPLE 1. Solve the equation

$$27x^6 + 46x^3 - 16 = 0.$$

Let $x^3 = y$. The equation then reduces to

$$27y^2 + 46y - 16 = 0,$$

which may be written in the form

$$(27y - 8)(y + 2) = 0.$$

Then $y = 8/27$, and $y = -2$. Hence $x^3 = 8/27$, and $x^3 = -2$. Therefore the roots of the equation in x are $x = 2/3$, and $x = \sqrt[3]{-2}$

EXAMPLE 2. Solve the equation

$$(2x^2 + 3x)^2 - 4(2x^2 + 3x) - 45 = 0.$$

Let $2x^2 + 3x = y$. Then

$$y^2 - 4y - 45 = 0,$$

or

$$(y - 9)(y + 5) = 0.$$

Then $y = 9$, and $y = -5$. Hence $2x^2 + 3x = 9$, and $2x^2 + 3x = -5$. Transposing the right member of each equation and solving for x, we obtain

$$x = \frac{-3 \pm \sqrt{9 + 72}}{4} = \frac{-3 \pm 9}{4},$$

or

$$x = \frac{3}{2} \quad \text{and} \quad x = -3,$$

and

$$x = \frac{-3 \pm \sqrt{9 - 40}}{4} = \frac{-3 \pm i\sqrt{31}}{4}.$$

EXAMPLE 3. Solve the equation

$$\frac{x^2 + 1}{x} - \frac{x}{x^2 + 1} = \frac{3}{2}.$$

Let

$$\frac{x^2 + 1}{x} = y.$$

Then

$$y - \frac{1}{y} = \frac{3}{2},$$

or

$$2y^2 - 3y - 2 = 0.$$

Whence

$$y = 2, \quad \text{and} \quad y = -\frac{1}{2}.$$

Hence

$$\frac{x^2+1}{x} = 2, \quad \text{and} \quad \frac{x^2+1}{x} = -\frac{1}{2}.$$

Solving these equations we obtain

$$x = 1, \quad \text{and} \quad x = 1,$$

and

$$x = \frac{-1 \pm i\sqrt{15}}{4}.$$

EXERCISES

Solve:

1. $x^4 - 25x^2 + 144 = 0.$
2. $3a^4 - 29a^2 + 18 = 0.$
3. $32x^{10} + 161x^5 + 5 = 0.$
4. $3x^{1/2} - 5x^{1/4} - 2 = 0.$
5. $a^{1/2} - 3a^{1/4} - 10 = 0.$
6. $4a^2(8a^2 + 3) - 5 = 0.$
7. $x^3 + 7x^{3/2} - 8 = 0.$
8. $5x^{-6/5} - 7x^{-3/5} + 2 = 0.$
9. $1 + 8x^{6/5} + 9x^{3/5} = 0.$
10. $x^{-2} - 2x^{-1} - 8 = 0.$
11. $\frac{x}{x^2-1} + \frac{x^2-1}{x} = \frac{13}{6}.$
12. $4a^{1/2} - 11a^{1/4} - 20 = 0.$
13. $x^{-6} + 19x^{-3} - 216 = 0.$
14. $27x^{3/2} - 26x^{3/4} - 1 = 0.$
15. $(x^2 - 3x)^2 + 3(x^2 - 3x) + 2 = 0.$
16. $(x^{2/3} - 2)(x^{4/3} - 4) + 12 = x^{2/3}(x^{2/3} - 1)^2.$
17. $(m^2 + 3)^2 - 5(m^2 + 3) = 14.$
18. $(a^2 - 5)^2 - 9(a^2 - 5) + 20 = 0.$
19. $x + 16 - 3\sqrt{x + 16} - 10 = 0.$
20. $x^2 + 6x + 2\sqrt{x^2 + 6x} - 24 = 0.$

21. $\left(x+\frac{3}{x}\right)^2+2\left(x+\frac{3}{x}\right)-15=0.$

22. $\frac{a^2-5a+1}{a^2-2a+2}-\frac{a^2-2a+2}{a^2-5a+1}+\frac{8}{3}=0.$

26. **Radical Equations.** An equation which contains the unknown number under a radical is frequently called a ***radical equation.*** It is generally necessary to square both members of such an equation in order to solve it. This process may introduce roots, called *extraneous roots*, which do not satisfy the original equation. Hence each root found should be tested and only those which satisfy the original equation should be retained.

Example 1. Solve the equation

$$3-\sqrt{x-1}=\sqrt{4x+5}.$$

Squaring both members of the equation, we find

$$9-6\sqrt{x-1}+x-1=4x+5.$$

Then

$$-6\sqrt{x-1}=3x-3,$$

or

$$-2\sqrt{x-1}=x-1.$$

Square again:

$$4(x-1)=x^2-2x+1,$$

or

$$x^2-6x+5=0.$$

Then

$$(x-1)(x-5)=0,$$

or

$$x=1 \quad \text{and} \quad x=5.$$

Check. Substituting 1 for x gives

$$3-0=\sqrt{9}, \quad \text{or} \quad 3=3.$$

Hence 1 is a root of the given equation.

Substituting 5 for x, we find

$$3 - \sqrt{4} = 25, \qquad \text{or} \qquad 3 - 2 = 5.$$

Hence 5 is an extraneous root and must be rejected.

NOTE. When both members of an equation are squared, extraneous roots are frequently introduced, even when the equation does not contain radicals. For example, the equation $x + 3 = 7$ has only one root, $x = 4$. But if we square both members we obtain, after reduction, $x^2 + 6x - 40 = 0$, or $(x - 4)(x + 10) = 0$, which contains the root, $x = -10$, which does not satisfy the original equation.

EXERCISES

Solve:

1. $2\sqrt{2x + 1} = 2x + 1.$
2. $a - 2 - 6\sqrt{a - 2} = 7.$
3. $2x + 7\sqrt{x} - 4 = 0.$
4. $\sqrt{5a - 6} - \sqrt{a - 2} = 2.$
5. $\sqrt{3 - x} + \sqrt{2 - x} = \sqrt{5 - 2x}.$
6. $\sqrt{5x - 1} - \sqrt{8 - 2x} = \sqrt{x - 1}.$
7. $\dfrac{\sqrt{a - 1} - \sqrt{a + 1}}{\sqrt{a - 1} + \sqrt{a + 1}} = \dfrac{4a - 7}{4}.$
8. $\dfrac{x - \sqrt{x + 1}}{x + \sqrt{x + 1}} = \dfrac{5}{11}.$
9. $\sqrt{7x + 8} - \sqrt{5x - 4} = 2.$
10. $\sqrt{7 - x} + \sqrt{3x + 10} = \sqrt{5 - 10x}.$
11. $\sqrt{2a + 6} - \sqrt{a + 4} = \sqrt{a - 4}.$
12. $\sqrt{x^2 + 2x - 3} - \sqrt{x^2 - 2x - 3} = \sqrt{2x^2 - 6}.$
13. $\sqrt{a - x} + \sqrt{b - x} = \sqrt{a + b - 2x}.$
14. $x^2 - 3x - 3 - 3\sqrt{x^2 - 3x + 7} = 0.$
15. $\dfrac{\sqrt{3x + 1} + \sqrt{3x}}{\sqrt{3x + 1} - \sqrt{3x}} = 2.$

16. $2\sqrt{3x+4} + 3\sqrt{3x+7} = \dfrac{8}{\sqrt{3x+4}}$.

17. $\sqrt{4a+3} + \sqrt{2a+3} = \sqrt{5a+1} + \sqrt{a+5}$.

18. $\sqrt{2a+3} - \sqrt{4a-3} + \sqrt{3a-5} - \sqrt{a+1} = 0$.

27. **Problems Involving Quadratic Equations.** If the solution of a problem involves a quadratic equation, there will always be two values of the unknown quantity. If either value fails to satisfy the conditions of the problem, it should be rejected. If it is necessary to reject both values, the problem is impossible.

EXAMPLE 1. Find two consecutive integers the sum of whose squares is 41.

Let

$$x = \text{one integer.}$$

Then

$$x + 1 = \text{the other integer,}$$

and

$$x^2 + (x+1)^2 = \text{the sum of the squares.}$$

By the conditions of the problem,

$$x^2 + (x+1)^2 = 41,$$

or

$$2x^2 + 2x - 40 = 0.$$

Whence

$$x^2 + x - 20 = 0,$$

and

$$x = 4 \quad \text{or} \quad x = -5.$$

Then

$$x + 1 = 5 \quad \text{or} \quad x + 1 = -4,$$

and the required integers are 4 and 5, or -5 and -4. It is evident that either set of integers will satisfy the conditions of the problem.

EXAMPLE 2. A merchant sold goods for \$50.69 and gained as many per cent as the goods cost dollars. What was the cost?

Let

$$x = \text{number of dollars the goods cost.}$$

Then

$$x = \text{per cent of gain,}$$

and

$$x \times \frac{x}{100}, \quad \text{or} \quad \frac{x^2}{100} = \text{number of dollars gained.}$$

Hence

$$50.69 - x = \frac{x^2}{100},$$

or

$$x^2 + 100x - 5069 = 0.$$

Solving for x gives $x = 37$ and $x = -137$.

The second value of x must be rejected since a negative answer has no meaning in this problem. The cost of the goods was, therefore, \$37.

Example 3. Find two integers whose sum is 18 and whose product is 25.

Let

$$x = \text{one integer.}$$

Then

$$18 - x = \text{the other integer,}$$

and

$$x(18 - x) = \text{the product.}$$

Hence

$$x(18 - x) = 25,$$

or

$$x^2 - 18x + 25 = 0.$$

Solving for x, we find

$$x = \frac{18 \pm \sqrt{224}}{2}.$$

Since neither of these values of x is an integer, they must both be rejected and the problem is impossible.

EXERCISES

1. The sum of a number and its reciprocal is 41/20. Find the number.

2. The square of the sum of two consecutive numbers exceeds the sum of their squares by 480. Find the numbers.

3. Find two consecutive even numbers, the sum of whose squares is 340.

4. One side of a rectangle is $3\frac{1}{2}$ feet longer than the other and the area is 200 square feet. Find its dimensions.

5. The difference between the hypotenuse and two sides of a right-angled triangle is 3 and 6, respectively. Find the sides.

6. The difference between the cubes of two consecutive numbers is 1519. Find the numbers.

7. The circumference of the hind wheel of a wagon is five feet more than that of the front wheel. If the hind wheel makes 150 fewer revolutions than the front wheel going one mile, find the circumference of each wheel.

8. If a train traveled five miles an hour faster, it would take two hours less to go 420 miles. Find the rate of the train.

9. In a company of 14 persons, men and women, the men spent \$50.00 and the women \$54.00. If each man spent \$4.00 more than each woman, how many men and how many women were in the company?

10. Two bodies move from the vertex of a right angle along its sides at the rate of 12 feet and 16 feet a second respectively. After how many seconds will they be 90 feet apart?

11. If a ball is thrown upwards with a velocity v_0, its distance d from the point of projection after a given time t is given by the formula

$$d = v_0 t - \tfrac{1}{2} g t^2,$$

where $g = 32$, and where the units of distance and time are feet and seconds. The speed at the time t is given by

$$v = v_0 - gt.$$

How long will it take a ball to fall from an airplane 900 feet high if it starts with no initial velocity?

12. If the ball is thrown downwards with a speed v_0, the formulas of Ex. 11 become

$$d = v_0 t + \tfrac{1}{2} g t^2,$$

$$v = v_0 + gt.$$

If a body is thrown downward with a speed of 50 feet a second from a distance of 225 feet, how long will it be before the body will strike the ground?

13. The edges of a cube are each increased in length two inches. It is found that the volume of the cube is thereby increased 98 cubic inches. What was the length of the edges of the cube?

14. The sum of two numbers is 30. One-third of their product plus 18 is equal to the square of the lesser number. Find the numbers.

15. The area of a square would be doubled if it were changed into a rectangle 6 inches longer and 4 inches wider. Determine its side.

16. The perimeter of a right triangle is 300 inches and the hypotenuse is 130 inches. Find the length of each side.

17. A merchant sold goods for $22.75, and lost as many per cent as the goods cost dollars. What was the cost?

18. The sum of the squares of four consecutive integers is 54. What are the integers?

19. A fast train runs 6 miles an hour faster than a slow train, and takes one hour and thirty minutes less to travel 340 miles. Find the rates of the trains.

20. Find the fraction which exceeds its reciprocal by 16/15.

21. A and B start from a certain point at the same time, A going North 4 miles per hour and B going East 3 miles per hour. In how many hours will they be 30 miles apart?

22. A man and a boy working together do a piece of work in 8 days. If we assume that the man could do it alone in 12 days less than the boy could do it, how long would it take each to do it?

23. A man walked 6 miles in a certain interval of time. Had the time been 30 minutes less, the rate would have been 2 miles per hour greater. Find the time and the rate.

24. A number is formed by two digits; five times the second digit exceeds the square of the first digit by 4. If three times the first digit is added to the number, the order of the digits is reversed. Find the number.

CHAPTER III

RATIO AND PROPORTION

28. **Ratio.** The *ratio* of one number a to a second number b is the quotient of a divided by b, that is, $\frac{a}{b}$. For convenience in printing this ratio is also expressed in the form $a:b$ or a/b. Hence all ratios of numbers are fractions and all fractions may be regarded as ratios.

Thus the ratio of 4 to 10 is 4/10, or 2/5.

It is frequently necessary to consider the ratio of two concrete magnitudes. In this case the ratio of the two magnitudes is the same as the ratio of the two numbers which express their measures in terms of a common unit.

Thus the ratio of two lines whose lengths are $6\frac{3}{4}$ inches and 9 inches, respectively, is $6\frac{3}{4}/9$, or 3/4.

The student makes use of ratios every time he measures anything. For example, when he says that the distance between two points is 36 feet he means that the ratio of this distance to a second (shorter) distance which he calls a "foot" is 36 : 1, or 36.

If the ratio can be expressed as a rational number, the magnitudes are *commensurable;* otherwise they are said to be *incommensurable.*

If the diameter of a circle is 1 yard, the ratio of the circumference to the diameter is $\pi : 1$. Since π is an irrational number, these lines are incommensurable.

29. **Proportion.** If two ratios are equal, their four terms are said to be ***in proportion.***

Thus $\frac{3}{5} = \frac{6}{10}$, $\frac{2}{7} = \frac{12}{42}$, and $\frac{9}{7} = \frac{27}{21}$ are proportions.

In any proportion the first and fourth terms are called the *extremes* and the second and third terms the *means*. If the means are equal, either mean is called a *mean proportional* between the first and fourth terms.

Thus, if $a/b = b/c$, then b is a mean proportional between a and c.

30. **Properties of Proportions.** A proportion is merely a statement of equality between two fractions. We may, therefore, by means of the fundamental operations of algebra transform this equation into a variety of forms, each of which will express some property of a proportion.

Several such properties are stated below. They can all be obtained by means of the usual operations on equations.

(1) If $a/b = c/d$, then $ad = bc$. Thus, *the product of the means is equal to the product of the extremes.*

(2) If $ad = bc$, then either $a/b = c/d$ or $c/d = a/b$. Thus, *if the product of two numbers equals the product of two others, either pair may be made the extremes and the other pair the means of a proportion.*

(3) If $a/b = c/d$, then $a/c = b/d$, or $d/b = c/a$. Thus, *in any proportion the means, or the extremes, may be interchanged.*

(4) If $a/b = c/d$, then $b/a = d/c$. Thus, *in any proportion the terms are in proportion by inversion.*

(5) If $a/b = b/c$, then $b = \sqrt{ac}$ Thus, *the mean proportional between two numbers is equal to the square root of their product.*

(6) Suppose $a/b = c/d$. Adding 1 to both members of the equation gives $(a + b)/b = (c + d)/d$. Thus, *in any proportion the terms are in proportion by "composition."*

(7) If $a/b = c/d$, then $(a - b)/b = (c - d)/d$. Thus, *in any proportion the terms are in proportion by "division."*

(8) If $a/b = c/d$, then $(a + b)/(a - b) = (c + d)/(c - d)$. Thus, *in any proportion the terms are in proportion by "composition and division."*

No undue importance should be attached to these forms, and they should not be memorized. Rather, it should be remembered that we can operate on any proportion by any rule which is correct for any equation.

EXAMPLE 1. If $a/b = c/d = e/f$, show that

$$\frac{a + c + e}{b + d + f} = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}.$$

Let $a/b = c/d = e/f = r$. Then $a = br$, $c = dr$, $e = fr$. Addition gives $a + c + e = (b + d + f)r$, or

$$\frac{a + c + e}{b + d + f} = r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}.$$

NOTE. This process may be extended to any number of equal ratios.

EXAMPLE 2. If $a/b = c/d$, show that

$$\frac{a^3 + b^3}{a^3 - 2ab^2} = \frac{c^3 + d^3}{c^3 - 2cd^2}.$$

Let $a/b = c/d = r$. Then $a = br$ and $c = dr$.
Thus

$$\frac{a^3 + b^3}{a^3 - 2ab^2} = \frac{b^3r^3 + b^3}{b^3r^3 - 2b^3r} = \frac{r^3 + 1}{r^3 - 2r},$$

and

$$\frac{c^3 + d^3}{c^3 - 2cd^2} = \frac{d^3r^3 + d^3}{d^3r^3 - 2d^3r} = \frac{r^3 + 1}{r^3 - 2r}.$$

Hence the two given fractions are equal.

EXAMPLE 3. Solve the equation

$$\frac{2x - 1}{2x + 4} = \frac{5x - 3}{5x + 9}.$$

Applying (7), that is, subtracting 1 from each side, we have

$$\frac{-5}{2x+4} = \frac{-12}{5x+9},$$

whence

$$25x + 45 = 24x + 48, \qquad \text{or} \qquad x = 3.$$

EXERCISES

Find the ratio of the following pairs of numbers:

1. 32, 50. **3.** $6\frac{1}{4}$, $3\frac{1}{2}$. **5.** 5.7, 19.

2. $4\frac{1}{2}$, 7. **4.** $7\frac{2}{3}$, $6\frac{4}{5}$. **6.** 2.4, 12.

7. Write the ratio of the length of the diagonal of a square to the length of one side.

8. What is the ratio of a to b if $\dfrac{2b+6a}{3a-b} = 10$?

9. Find the third term of a proportion whose first, second and fourth terms are 1/3, 2/5, and 3/4, respectively.

Find a mean proportional between

10. 12.1 and 1.6 **11.** $\dfrac{a+b}{a-b}$ and $\dfrac{a^2-b^2}{a^2b^2}$.

12. What number must be added to each of the numbers 12, 17, 4 and 7 to make them proportional?

13. Find two numbers proportional to 2 and 3 whose sum is 70.

14. What number must be added to each term of the ratio 2/3 so that the resulting ratio may be 12/15?

15. Find two numbers such that if 2 be added to the first and 7 to the second, the sums will be in the ratio 1/2, and if 5 be subtracted from the first and 1 from the second, the remainders will be in the ratio 1/4.

16. The sum of four numbers in proportion is 70. The sum of the extremes exceeds the sum of the means by 8, and the last term is 12 times the first term. Find the numbers.

17. A wheel 60 inches in diameter, making 150 revolutions per minute, is belted to another wheel 30 inches in diameter. Find the speed of the smaller wheel.

18. A passenger observes that a train passes him, moving in the opposite direction, in 4 seconds; while if it had been moving in the same direction, it would have passed him in 11 seconds. Compare the rates of the trains.

19. The diameters of the earth and the moon are in the ratio 80/21. The volumes of two spheres are proportional to the cubes of their respective diameters. Find the ratio of the volumes of the earth and the moon.

If $a/b = b/c$, show that

20. $a/c = a^2/b^2$.

21. $\dfrac{a}{a+b} = \dfrac{a-b}{a-c}$.

If $a/b = c/d$ and $e/f = g/h$, show that

22. $ae/bf = cg/dh$.

23. $\dfrac{ae-bf}{ae+bf} = \dfrac{cg-dh}{cg+dh}$.

If $a/b = c/d$, show that

24. $ma/nb = mc/nd$.

25. $a^n/b^n = c^n/d^n$.

26. $\dfrac{4a^2-3ab}{3ab+2b^2} = \dfrac{4c^2-3cd}{3cd+2d^2}$.

27. $\dfrac{a-c}{b-d} = \dfrac{\sqrt{a^2+c^2}}{\sqrt{b^2+d^2}}$.

28. $\dfrac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} = \dfrac{\sqrt[3]{a^3+b^3}}{\sqrt[3]{c^3+d^3}}$.

If $a/b = b/c = c/d$, show that

29. $a/d = a^3/b^3$.

30. $(b+c)(b+d) = (a+c)(c+d)$.

If $a/b = c/d = e/f$, show that

31. $\dfrac{a+c-e}{b+d-f} = \dfrac{a}{b}$.

32. $\dfrac{a^2-c^2+e^2}{b^2-d^2+f^2} = \dfrac{c^2}{d^2}$.

Solve the following equations:

33. $\dfrac{2x+3}{2x-3} = \dfrac{2b-3a}{2b+3a}$.

34. $\dfrac{x^2+3x-1}{x^2-3x+1} = \dfrac{x^2+2x+1}{x^2-2x-1}$.

CHAPTER IV

THE PROGRESSIONS

I. Arithmetic Progression

31. **Definition.** An *arithmetic progression* is a sequence of numbers, called *terms*, each of which is derived from the preceding by adding to it a constant number, called the *common difference.*

Thus 2, 5, 8, 11, 14, $\cdots$, and 3, 1, -1, -3, -5, $\cdots$, are arithmetic progressions whose common differences are 3 and -2 respectively.

An arithmetic progression may be extended to any number of terms. This is indicated by dots as above. For brevity we shall denote an arithmetic progression by the abbreviation A. P.

The first term of an A. P. is usually denoted by a and the common difference by d. Hence any A. P. may be written in the form

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d,\ \cdots.$$

32. **The *n*th Term of an A. P.** If an A. P. be extended to any number of terms, it is obvious that the coefficient of d in any given term will be 1 less than the number of that term. Hence the last term of an $A.\ P.$ of n terms is given by the formula

$$l = a + (n-1)d. \tag{1}$$

Example. Find the 10th term of the A. P. 2, 5, 8, 11, $\cdots$.

Here $a = 2$, $d = 3$, $n = 10$. Hence $l = 2 + 9 \times 3 = 29$.

The student should verify this result by extending the series to the tenth term.

33. **The Sum of the First n Terms of an A. P.** If s denotes the sum of the first n terms, it is evident that

$$s = a + (a + d) + (a + 2d) + \cdots + (l - 2d) + (l - d) + l.$$

Writing the terms in reverse order gives

$$s = l + (l - d) + (l - 2d) + \cdots + (a + 2d) + (a + d) + a.$$

Add these equations term by term.

$$2s = (a + l) + (a + l) + (a + l) + \cdots + (a + l) + (a + l) + (a + l) = n(a + l).$$

Hence

$$s = \frac{n}{2}(a + l). \tag{2}$$

Replacing l by its value $a + (n - 1)d$ gives

$$s = \frac{n}{2}[2a + (n - 1)d]. \tag{3}$$

If any three of the five elements a, d, n, l, s of an A. P. are given, the other two may be found by means of the equations (1), (2), and (3).

Example 1. Find the sum of the first 8 terms of the A. P. $1, 3, 5, 7, \cdots$.

Here $a = 1$, $d = 2$, and $n = 8$. Hence, from equation (3),

$$s = \frac{8}{2}(2 + 7 \times 2) = 4(2 + 14) = 64.$$

Example 2. Given $a = 8$, $n = 25$, $s = -1000$; find d and l.

We have from (3),

$$-1000 = \frac{25}{2}(16 + 24d),$$

or

$$-40 = 8 + 12d, \qquad \text{and} \qquad d = -4.$$

Then, from (1),

$$l = 8 + 24(-4) = -88.$$

EXAMPLE 3. Given $d = -2$, $l = -10$, $s = -24$; find a and n.

We have, from (1),

$$-10 = a + (n-1)(-2), \qquad \text{or} \qquad a = 2n - 12.$$

Then from (2),

$$-24 = \frac{n}{2}(2n - 12 - 10), \qquad \text{or} \qquad n^2 - 11n + 24 = 0.$$

Whence $n = 3$ and $n = 8$.
Then $a = 2 \times 3 - 12 = -6$, and $a = 2 \times 8 - 12 = 4$.
Hence the required A. P. is either $-6, -8, -10$, or $4, 2, 0, -2, -4, -6, -8, -10$. In either case the sum is -24.

EXERCISES

In each of the following arithmetic progressions find the last term and the sum of the terms.

1. $1, 3, 5, 7, \cdots$ to 20 terms.
2. $7, 4, 1, -2, \cdots$ to 12 terms.
3. $-5, -1, 3, 7, \cdots$ to 15 terms.
4. $-3/10, -5/6, -41/30, \cdots$ to 25 terms.
5. $2x, 7x, 12x, 17x, \cdots$ to 18 terms.
6. $3a - 2b, 4a - b, 5a, 6a + b, \cdots$ to 10 terms.
7. $\frac{a-b}{a+b}, \frac{2a-3b}{a+b}, \frac{3a-5b}{a+b}, \cdots$ to 12 terms.
8. Given $a = -69$, $n = 22$, $l = 36$; find d and s.
9. Given $d = 3$, $l = 31$, $n = 10$; find a and s.
10. Given $a = -31/6$, $d = 1/3$, $s = -77/2$; find n and l
11. Given $a = x$, $l = 13x$, $s = 49x$; find n and d.

12. Given a, l, and n; derive the formula for d.

13. Given l, n, and s; derive the formulas for a and d.

14. A marble rolls down an inclined plane, passing over distances 8 feet, 24 feet, 40 feet, etc., in successive seconds. How far will it roll in 10 seconds?

15. A man is paying for a home on the installment plan. His payments increase twenty cents each month and for the first three months are \$40.00, \$40.20, and \$40.40. How much will he pay in five years?

34. **Arithmetic Means.** It is always possible to insert a set of numbers between two given numbers so that the result shall be an A. P. of a given number of terms.

Thus, if the given numbers are 3 and 15, we may write an A. P. of three terms, 3, 9, 15; four terms, 3, 7, 11, 15; five terms, 3, 6, 9, 12, 15; six terms, 3, $\frac{27}{5}$, $\frac{39}{5}$, $\frac{51}{5}$, $\frac{63}{5}$, 15; seven terms, 3, 5, 7, 9, 11, 13, 15; etc.

The terms of an A. P. that lie between two given terms are called the ***arithmetic means*** between those terms. If the progression consists of only three terms, the middle term is called the *arithmetic mean* of the other two.

Thus 9 is the arithmetic mean of 3 and 15.

If m denotes the arithmetic mean between the numbers p and q, then $d = m - p = q - m$. Whence

$$m = \frac{p + q}{2}.$$

Thus, *the arithmetic mean of two numbers is equal to one-half their sum.*

The arithmetic mean of two numbers is also called their *average.*

EXAMPLE 1. Insert six arithmetic means between -3 and 25.

Here $a = -3$, $n = 8$, and $l = 25$. Then, from (1),

$$25 = -3 + 7d, \qquad \text{or} \qquad d = 4.$$

Hence the A. P. is $-3, 1, 5, 9, 13, 17, 21, 25$.

EXAMPLE 2. Write the A. P. whose third and eighth terms are -5 and -45, respectively.

We have, from (1),

$$a + 7d = -45,$$

and

$$a + 2d = -5.$$

Solving this system of equations gives $a = 11$ and $d = -8$. Then the progression is $11, 3, -5, -13, -21, -29, -37, -45$.

EXERCISES

1. Find the sum of an A. P. of sixteen terms, the first of which is -69 and the last of which is 36.

2. If the A. P., $2, 4, 6, 8, \cdots$, be extended until the last term is 48, what is the sum of the terms?

3. Insert eight arithmetic means between -9 and 36.

4. Insert six arithmetic means between $3/4$ and $1/6$.

5. The sum of fifteen terms of an A. P. is 600, and the common difference is 5. Find the first term.

6. How many terms of the series $3, 6, 9, 12, \cdots$, must be taken so that the sum shall be 2583?

7. The last term of an A. P. of nineteen terms is 147 and the common difference is 8. Find the first term.

8. The sum of the terms of an A. P. whose last term is $23/12$ is $-58\,3$. If the common difference is $5/12$, what is the first term?

9. Write the A. P. whose fourth and tenth terms are -13 and 5, respectively.

10. The 12th, 85th, and last terms of an A. P. are 38, 257, and 395, respectively; find the number of terms.

11. Find the sum of all the odd numbers between 200 and 300.

12. How many arithmetic means must be inserted between the two numbers -6 and 34 so that the sum of the terms of the A. P. shall be 154?

13. A body falls 16.1 feet in the first second, 48.3 feet in the next, and 80.5 feet in the third. How far will it fall in 10 seconds?

14. If a ball be dropped from the top of the Washington Monument, which is 555.4 feet high, how long will it take it to reach the ground?

15. How many strokes does a common clock make in a day?

16. A wheel is turning 800 times a minute when a brake is applied which reduces its speed by 50 revolutions per minute. How many revolutions does the wheel make before stopping?

17. Find the sum of all positive integers less than 500 which are multiples of 9.

18. Find the number of arithmetic means between 1 and 25, if the first mean is to the last mean as 2 : 11.

19. Find the sum of the terms of an A. P. of nine terms, in which the middle term is 15.

20. If a man deposits 25 cents in a savings account during the first week in January, 50 cents the second week, 75 cents the third week, and so on, how much will he deposit in a year?

21. The sum of five numbers in A. P. is 10 and the sum of their squares is 180. Find the numbers.

22. The sum of the first 10 terms of an A. P. is to the sum of the first 5 terms as 13 to 4. Find the ratio of the first term to the common difference.

23. A man makes a note for \$3000 at 10% and agrees to pay \$30.00 and interest monthly. How much will he have paid when he has freed himself of the debt?

24. Show that the sum of the first n odd numbers is n^2.

25. Show that the sum of the first n terms of an A. P. is n times the arithmetic mean of the first and last terms.

26. Show that, if every term of an A. P. be multiplied, or divided, by the same number, the resulting numbers will form an A. P.

27. In a potato race, twenty potatoes are placed in a line six feet apart. Each contestant starts from a position in line with the potatoes and ten feet from the first one and picks them up in succession, returning with each to his original position. How far does he travel?

28. The three digits of a number are in A. P. Their sum is 12, and the last digit exceeds the first by an amount equal to the middle digit. Find the number.

29. In a lottery fifty people draw tickets numbered 1, 2, 3, 4, $\cdots$ up to 50. If the price of a ticket in cents is the number on the ticket, how much money is paid in?

30. If a man sold a gold watch using the same system as in Exercise 29 with tickets numbered from 1 to 100, how much did he receive for the watch?

31. A tourist starts from a certain place and travels 10 miles the first hour, 15 miles the second hour, 20 miles the third hour, and so on. After he has been gone two hours and fifteen minutes another tourist sets out from the same place and travels 40 miles an hour in the same direction. After how many hours are the tourists together?

II. Geometric Progression

35. **Definition.** A ***geometric progression*** (abbreviated G. P.) is a sequence of numbers, called *terms*, each of which is derived from the preceding by multiplying it by a constant number, called the *common ratio.*

Thus

$$4, 12, 36, 108, \cdots,$$

and

$$4, -2, 1, -\tfrac{1}{2}, \cdots,$$

are geometric progressions whose common ratios are 3 and $-\frac{1}{2}$, respectively.

The first term of a G. P. is usually denoted by a and the common ratio by r. Hence any G. P. may be written in the from

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,\ \cdots.$$

36. **The *n*th Term of a G. P.** If a G. P. be extended to any number of terms, it is obvious that the exponent of r in any given term will be 1 less than the number of that term. Hence the last term of a G. P. of n terms is

$$l = ar^{n-1}. \tag{1}$$

EXAMPLE. Find the eighth term of the G. P. $4, 6, 9, \frac{27}{2}, \cdots$.

Here $a = 4$, $r = \frac{3}{2}$, $n = 8$. Hence

$$l = 4\left(\frac{3}{2}\right)^7 = \frac{4 \times 2187}{128} = \frac{2187}{32}.$$

37. **The Sum of the First *n* Terms of a G. P.** If s denotes the sum of the first n terms, it is evident that

$$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1}.$$

Multiplying both members of this equation by r gives

$$rs = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n.$$

If these equations be subtracted, the result is

$$s - rs = a - ar^n.$$

Then, if $r \neq 1$,

$$s = \frac{a - ar^n}{1 - r}. \tag{2}$$

If $r = 1$, the G. P. is $a, a, a, a, \cdots$ and the sum is na. Since $ar^n = r \times ar^{n-1} = rl$, this equation may be written

$$s = \frac{a - rl}{1 - r}. \tag{3}$$

As in § 33, if any three of the five elements a, r, n, l, s of a G. P. are given, the other two may be found by means of

equations (1), (2), and (3). However, this operation usually involves the solution of equations of types which have not yet been considered. In this chapter only those cases are included where the equations may be solved by inspection.

EXAMPLE 1. Find the sum of the first seven terms of the G. P. 2, 6, 18, $\cdots$.

Here $a = 2$, $r = 3$, and $n = 7$. Hence, from equation (2),

$$s = \frac{2 - 2 \times 3^7}{1 - 3} = \frac{2 - 2 \times 2187}{-2} = \frac{2 - 4374}{-2} = \frac{-4372}{-2} = 2186.$$

EXAMPLE 2. Given $a = 3$, $n = 7$, $l = 192$; find r and s.

We have, from (1), $192 = 3r^6$; whence $r^6 = 64$, and $r = \pm 2$. Let us now substitute these values of r in (3).

If $r = 2$,

$$s = \frac{3 - 2 \times 192}{1 - 2} = \frac{3 - 384}{-1} = 381.$$

If $r = -2$,

$$s = \frac{3 - (-2) \times 192}{1 - (-2)} = \frac{3 + 384}{3} = 129.$$

Hence the solution is $r = 2$, $s = 381$; or $r = -2$, $s = 129$.

EXAMPLE 3. Given $r = -2/3$, $n = 7$, $s = 154\frac{1}{3}$; find a and l.

We have

$$a - ar^n = a(1 - r^n) = a\left(1 + \frac{128}{2187}\right) = \frac{2315a}{2187},$$

and

$$1 - r = 1 - \left(-\frac{2}{3}\right) = \frac{5}{3}.$$

Hence, from (2),

$$\frac{463}{3} = \frac{2315a}{2187} \div \frac{5}{3} = \frac{463a}{729}, \qquad \text{or} \qquad a = 243.$$

Then from (1),

$$l = 243\left(-\frac{2}{3}\right)^6 = \frac{243 \times 64}{729} = \frac{64}{3}.$$

EXERCISES

In each of the following geometric progressions find the last term and the sum.

1. 3, 6, 12, 24, $\cdots$ to 10 terms.
2. 4, -8, 16, -32, $\cdots$ to 7 terms.
3. -2, 6, -18, 54, $\cdots$ to 6 terms.
4. 1, 2, 4, 8, $\cdots$ to 9 terms.
5. 1/2, 1/4, 1/8, $\cdots$ to 11 terms.
6. $-3/8$, 3/2, -6, 24, $\cdots$ to 8 terms.
7. Given r, l, and s; derive the formula for a.
8. Given $r = 2$, $n = 8$, $s = 1020$; find a and l.
9. Given $a = 6$, $r = 4/5$, $s = \dfrac{2214}{125}$; find l and n.
10. Given $a = 13$, $l = 208$, $s = 403$; find r and n.
11. Given $a = 7$, $r = 10$, $l = 70000$; find n and s.
12. If the first term is 3 and the ratio 2, what term will be 384?

38. **Geometric Means.** The terms of a G. P. that lie between two given terms are called the ***geometric means*** between these terms. If the progression consists of only three terms, the middle term is called the *geometric mean* of the other two.

As in § 34 it is possible to insert more than one set of geometric means between two given numbers, say 2 and 128. Thus we may write a G. P. of three terms, 2, 16, 128; four terms, 2, 8, 32, 128; seven terms, 2, 4, 8, 16, 32, 64, 128; etc.

If m denotes the geometric mean between the numbers p and q, then $p/m = m/q$, or $m^2 = pq$. Whence

$$m = \pm\sqrt{pq}.$$

Thus, *the geometric mean of two numbers is equal to the square root of their product.*

EXAMPLE 1. Insert five geometric means between 3 and $\frac{64}{243}$.

Here $a = 3$, $n = 7$, and $l = \frac{64}{243}$.

Then from (1),

$$\frac{64}{243} = 3r^6, \qquad \text{or} \qquad r^6 = \frac{64}{729} = \left(\pm\frac{2}{3}\right)^6.$$

Hence $r = \pm\frac{2}{3}$ and the G. P. is either

$$3, 2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \frac{32}{81}, \frac{64}{243}, \quad \text{or} \quad 3, -2, \frac{4}{3}, -\frac{8}{9}, \frac{16}{27}, -\frac{32}{81}, \frac{64}{243}.$$

EXAMPLE 2. Write the G. P. whose third and sixth terms are 6 and $\frac{81}{4}$, respectively.

We have, from (1),

$$ar^2 = 6, \qquad \text{and} \qquad ar^5 = \frac{81}{4}.$$

Division gives

$$r^3 = \frac{81}{24} = \frac{27}{8}, \qquad \text{or} \qquad r = \frac{3}{2}.$$

Then $a\left(\frac{3}{2}\right)^2 = 6$, or $a = \frac{8}{3}$.

Hence the required G. P. is $\frac{8}{3}, 4, 6, 9, \frac{27}{2}, \frac{81}{4}$.

39. **Infinite Geometric Progressions.** A progression in which the value of n is not stated but is allowed to increase without limit, is called an ***infinite progression.*** If the sum of the first n terms of a progression approaches nearer and nearer a fixed finite number, as n increases without limit, this finite number is called the sum *to infinity* of the progression.

It is evident that the numerical value of the sum of the first n terms of any arithmetic progression may be made greater than any preassigned number however large by

making n sufficiently large. That is, it is not possible to find a finite number which will represent the sum to infinity of any arithmetic progression.

The same is true of any geometric progression where the common ratio r is numerically equal to or greater than 1. Hence the only progression which could possibly have a sum to infinity is a geometric progression where the numerical value of r is less than 1, that is, where r is between -1 and $+1$.

Let us consider the G. P

$$1, \quad 1/2, \quad 1/4, \quad 1/8, \quad 1/16, \quad 1/32, \quad \cdots,$$

where $a = 1$ and $r = 1/2$.

The sum of the first n terms of this G. P. may be exhibited graphically as follows:

Fig. 2

Suppose we undertake a journey of two miles, going 1 mile the first hour, 1/2 mile the second hour, 1/4 mile the third, 1/8 mile the fourth, and so on, each hour going just one-half the distance we went the hour before. If this is kept up indefinitely, we evidently come as near as we please to our destination. Thus the sum to infinity of the G. P. 1, 1/2, 1/4, 1/8, $\cdots$ is 2.

Let us now consider the sum to infinity of *any* infinite geometric progression,

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad \cdots,$$

where the numerical value of r is less than 1. We have, from (2),

$$s = \frac{a - ar^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

The expression r^n may be made as small as one pleases

by taking n larger and larger. Hence the second fraction approaches 0 and the formula for the sum to infinity of a decreasing geometric progression is

$$s_\infty = \frac{a}{1 - r}, \tag{4}$$

where s_∞ means the sum to infinity.

Example 1. Find the sum to infinity of the G. P.,

$$2, \quad \frac{1}{2}, \quad \frac{1}{8}, \quad \frac{1}{32}, \quad \frac{1}{128}, \quad \cdots.$$

Here $a = 2$ and $r = \frac{1}{4}$. Hence

$$s_\infty = \frac{2}{1 - \frac{1}{4}} = \frac{2}{\frac{3}{4}} = 2\tfrac{2}{3}.$$

Example 2. Find the value of the repeating decimal $0.285151\cdots$.

We have

$$0.285151\cdots = 0.28 + .0051 + .000051 + .00000051 + \cdots.$$

The terms after the first form a G. P., in which $a = .0051$ and $r = .01$. Since r is less than 1 we may use (4).

Thus

$$s_\infty = \frac{.0051}{1 - .01} = \frac{.0051}{.99} = \frac{51}{9900} = \frac{17}{3300},$$

and

$$0.285151\cdots = \frac{28}{100} + \frac{17}{3300} = \frac{941}{3300}.$$

Note. It may be shown that every rational number (§ 1), that is, every number which may be written as the quotient of two integers, may be expressed as a *repeating* decimal, while every irrational number when expressed as a decimal gives rise to a *non-repeating* decimal. A terminating decimal is here considered as a repeating decimal where all digits after a given point are zero. The proof of this statement is beyond the scope of this book.

EXERCISES

Find the geometric mean between:

1. $\frac{15}{8}$ and $\frac{10}{3}$. 2. $x^2 - 6xy + 9y^2$ and $4x^2 + 4xy + y^2$.

3. Insert five geometric means between $3\frac{5}{9}$ and $40\frac{1}{2}$.

4. Insert six geometric means between 12 and 3/32.

5. Insert seven geometric means between 81 and 1/81.

Find the sum to infinity:

6. 9, 6, 4, 8/3, $\cdots$.

7. 4, $-$ 8/3, 16/9, $-$ 32/27, $\cdots$.

8. 12, 4, 4/3, 4/9, $\cdots$.

9. $-$ 2, 5/3, $-$ 25/18, 125/108, $\cdots$.

10. 4/5, 2/15, 1/45, 1/270, $\cdots$.

11. .8, $-$ .4, .2, $-$ .1, $\cdots$.

12. $-$ 7/8, $-$ 7/12, $-$ 7/18, $-$ 7/27, $\cdots$.

13. 18, 4, 8/9, 16/81, $\cdots$.

14. 16, $-$ 12, 9, $-$ 27/4, $\cdots$.

15. $-$ 1/50, 1/60, $-$ 1/72, 5/432, $\cdots$.

Find the value of:

16. 2.333$\cdots$. 19. 0.4747$\cdots$. 22. 0.2134134$\cdots$.

17. 4.999$\cdots$. 20. 0.231212$\cdots$. 23. 0.543636$\cdots$.

18. 0.2727$\cdots$. 21. 0.341341$\cdots$. 24. 0.077777$\cdots$.

25. Find four numbers in G. P. such that the sum of the first and fourth is 195, and of the second and third, 60.

26. Find an A. P. whose first term is 2, and whose first, second, and fifth terms form a G. P.

27. The product of three terms of a G. P. is 64. Find the second term.

28. Find six numbers in G. P. such that the sum of the first and fourth shall be 28, and of the third and sixth, 252.

29. Find three numbers in G. P. such that their sum shall be 21 and the sum of their squares 189.

30. The sum of the first four terms of a G. P. is 15, and the sum of the terms from the second to the fifth inclusive is 30. What are the first four terms of the series?

31. If you take half of a line 8 inches long, and half of what is left, and so on, what is the limit of the sum of these parts?

32. If a rubber ball rebounds to three-fifths of the height from which it has fallen, how great a distance will it traverse if dropped from a height of 40 feet?

33. A number consists of three digits in G. P. The sum of the digits is 14; and if 594 be added to the number the digits will be reversed. Find the number.

34. If the numbers a, b, c, d form a G. P., show that

$$(a - d)^2 = (b - c)^2 + (c - a)^2 + (d - b)^2.$$

35. A man who saved every year 50% more than in the preceding year, had saved \$6500 in four years. How much did he save the first year?

36. Solve the equation $x^3 - 1 = 0$ and then show that its roots form a G. P.

37. If a sequence of numbers form a geometric progression, will the squares of these numbers likewise form a G. P.? the cubes? the square roots? Answer the same questions for an A. P.

38. Show that the reciprocals of the terms of a G. P. form another G. P. Is this true for an A. P.?

39. A person contributes one cent and sends letters to two friends, asking each to contribute one cent to a certain charity and to write a similar letter to each of two friends, each of whom is to write two letters—and so on until fifteen sets of letters have been written. If all respond, how much money will the charity receive?

III. Harmonic Progression

40. **Definition.** We have seen (Ex. 38, above) that the reciprocals of the terms of a geometric progression form another geometric progression. If, however, we invert each of the terms of an arithmetic progression, we obtain a sequence of numbers which form neither an arithmetic nor a geometric progression. Such a sequence is called a ***harmonic progression.*** That is, a harmonic progression is a

sequence of numbers whose reciprocals form an arithmetic progression.

Thus 1/2, 1/5, 1/8, 1/11, $\cdots$, and 1/3, 1, -1, $-1/3$, $-1/5$, $\cdots$, are harmonic progressions.

In order to find the nth term of an H. P. we transform it into an A. P., find the nth term of the A. P., and then invert that term.

EXAMPLE. Find the 11th term of 1/9, 1/2, $-1/5$, $-1/12$, $\cdots$.

The corresponding A. P. is 9, 2, -5, -12, $\cdots$.
The 11th term is $l = 9 + 10 \times (-7) = -61$.
Hence the 11th term of the H. P. is $-1/61$.

There is no general method for finding the sum of the first n terms of a harmonic progression.

41. **Harmonic Means.** The terms of an H. P. which lie between two given terms are called the ***harmonic means*** between these terms. If the progression consists of only three terms, the middle term is called the *harmonic mean* of the other two.

If m denotes the harmonic mean between the numbers p and q, then $1/m$ is the arithmetic mean between $1/p$ and $1/q$. Then

$$\frac{1}{m} = \frac{\frac{1}{p} + \frac{1}{q}}{2} = \frac{p+q}{2pq},$$

and

$$m = \frac{2pq}{p+q}.$$

EXAMPLE. Insert three harmonic means between 4 and 12.

We must insert three arithmetic means between 1/4 and 1/12.
Here $a = 1/4$, $n = 5$, and $l = 1/12$.
Hence $1/12 = 1/4 + 4d$, or $d = -1/24$.
Thus the A. P. is 1/4, 5/24, 1/6, 1/8, 1/12, and the H. P. is 4, 24/5, 6, 8, 12.

EXERCISES

In each of the following harmonic progressions find the last term:

1. $\frac{3}{2}, \frac{6}{7}, \frac{6}{10}, \cdots$ to 10 terms.

2. $4, -12, -\frac{12}{5}, \cdots$ to 8 terms.

3. $-\frac{1}{3}, 1, \frac{1}{5}, \cdots$ to 15 terms.

4. $-\frac{1}{6}, -\frac{4}{27}, -\frac{2}{15}, \cdots$ to 18 terms.

Find the harmonic mean between:

5. $\frac{1}{9}$ and $\frac{1}{27}$.

6. $\frac{1}{a-b}$ and $\frac{1}{a+b}$.

7. Insert four harmonic means between 3 and 18.

8. Insert six harmonic means between $-\frac{2}{5}$ and $\frac{2}{9}$.

9. The arithmetic mean between two numbers is 6, and the harmonic mean is 35/6. What are the numbers?

10. The harmonic mean between two numbers exceeds their arithmetic mean by 7, and one number is three times the other. Find the numbers.

11. Three quantities are in H. P. Show that if half the middle term be subtracted from each the remainders are in G. P.

12. If a^2, b^2, c^2 be in A. P., show that $b + c$, $c + a$, $a + b$ are in H. P.

13. If a, b, c are in H. P., show that a, $a - c$, $a - b$ are in H. P.

14. If A, G, H denote the arithmetic, geometric, and harmonic means between a and b, show that G is the geometric mean between A and H.

15. Show that if H is the harmonic mean between a and b, then

$$\frac{1}{H-a} + \frac{1}{H-b} = \frac{1}{a} + \frac{1}{b}.$$

CHAPTER V

LOGARITHMS

42. **Introduction.** In the preceding chapters all of the exponents have been either integers or fractions (positive or negative), fractional and negative exponents having been defined (§ 14) in such a way that they obey all the laws of operation.

In what follows we shall assume that the exponents are *real*, but not necessarily *rational*, numbers. It may be shown that irrational exponents obey all the laws of operation, although the proof is beyond the scope of this book.

The meaning of an irrational exponent is not difficult to understand. We have seen (§ 1) that the number $\sqrt{2}$ lies between 1 and 1.5 and that a sequence of rational numbers, 1, 1.4, 1.41, 1.414, 1.4142, $\cdots$, may be found which will approximate its true value to any degree of accuracy. Consequently the number $4^{\sqrt{2}}$ lies between 4^1 $(= 4)$ and $4^{1.5}$ $(= 4^{3/2} = 8)$ and a very close approximation to its value is $4^{1.414}$ $(= 4^{1414/1000} = \sqrt[1000]{4^{1414}})$.

The result of raising a number to an irrational power is, of course, an irrational number whose value may be only approximated. However, by the use of the tables which will presently be explained, the value of such a number may be quickly approximated with a high degree of accuracy.

43. **Definition.** Suppose we wish to multiply 64 by 512. We may write $64 = 2^6$ and $512 = 2^9$; then $64 \times 512 = 2^6 \times 2^9 = 2^{15}$. Now suppose we have a table from which we may obtain all the powers of 2, and we find that

$2^{15} = 32768$. We may then say that $64 \times 512 = 32768$, without actually multiplying the two numbers.

Again, let us suppose that we have a simple method of expressing any given number as a power of 10 and that, for example, $21.6 = 10^{1.3345}$ and $63.2 = 10^{1.8007}$. Then $21.6 \times 63.2 = 10^{1.3345} \times 10^{1.8007} = 10^{3.1352}$. If we also have a simple method of determining that $10^{3.1352} = 1365$ approximately, we have found that the product of 21.6 and 63.2 is approximately equal to 1365, without actually performing the multiplication.

Let us assume that every positive number may be written as some power of another positive number, which we shall call the *base*. The exponent of the power to which the base must be raised to produce the number is called the ***logarithm*** of the number to the given base. Thus if

$$a^m = N, \tag{1}$$

then m is the logarithm of N to the base a. This is written

$$m = \log_a N. \tag{2}$$

In the above examples $6 = \log_2 64$, $15 = \log_2 32768$, $1.3345 = \log_{10} 21.6$, $3.1352 = \log_{10} 1365$, etc.

It should be observed that a logarithm is nothing but an exponent and that equations (1) and (2) are simply two different methods of expressing one and the same thing. Equation (1) states a fact in *exponential form* and equation (2) states the same fact in *logarithmic form*.

Logarithms were first introduced by the Scotch mathematician, John Napier (1550–1617). Later, the irrational number e ($= 2.7182 +$) was used as a base. Their application to numerical calculations was very much simplified by the English mathematician, Henry Briggs (1556–1630), who used 10 as the base and computed the first table of logarithms.

EXERCISES

1. Using 2 as the base, state the logarithm of each of the following numbers:

32, 128, 2048, 16, 256, 16384, 2, 1/2, 1/16.

2. State the *meaning* and then the *value* of each of the following expressions:

$\log_2 64$, $\log_{10} 100$, $\log_5 125$, $\log_7 343$, $\log_9 3$, $\log_4 4$.

3. Write the following equations in logarithmic form:

$3^4 = 81$, $5^4 = 625$, $6^1 = 6$,
$10^3 = 1000$, $8^{2/3} = 4$, $3^0 = 1$,
$4^5 = 1024$, $27^{-2/3} = 1/9$, $10^{-2} = .01$.

4. Why can 1 not be used as base for a system of logarithms?

5. Why can a negative number not be used as a base for a system of logarithms?

6. Solve the following equations:

$\log_3 x = 4$, $\log_x 27 = -3$, $\log_8 x = -2/3$,
$\log_2 x = 6$, $\log_{25} x = -1/2$, $\log_x 64 = 3$.

Which of the following statements are true and which are false?

7. $\log_4 64 = 3$.
8. $\log_3 125 = 5$.
9. $\log_9 3 = -1/2$.
10. $\log_2 16 = 4$.
11. $\log_8 4 = 2/3$.
12. $\log_3 81 = 3$.

Verify the following:

13. $\log_2 16 = \log_2 8 + \log_2 2$.

14. $\log_2 1024 = \log_2 16 + \log_2 8 + \log_2 4 + \log_2 2$.

15. $\log_3 243 = \log_3 27 + \log_3 9$.

16. $\log_4 64 = \log_4 32 + \log_4 8 - \log_4 4$.

17. Using 2 as the base show that

$$\log 4096 = \log 512 + \log 8.$$

Is this statement still true if the base is 4? If the base is 8?

44. **Properties of Logarithms.** Since logarithms are simply exponents, each of the laws of exponents will lead to a property of logarithms. We shall state and prove several of these properties, the base a being, in each case, a positive real number greater than 1.

(1) *The logarithm of the base is 1.*

For $a^1 = a$, and hence $\log_a a = 1$.

(2) *The logarithm of 1 is 0.*

For $a^0 = 1$, and hence $\log_a 1 = 0$.

(3) *The logarithm of a number greater than* 1 *is positive.*

Consider the equation $a^x = N$, where $N > 1$. Suppose x is negative and equal to $-y$, where y is positive. Then $a^x = 1/a^y = N$, or $a^y \times N = 1$, which is impossible since $a > 1$ and $N > 1$. Hence $x\ (= \log_a N)$ must be positive.

(4) *The logarithm of a number between 0 and 1 is negative.*

The proof of this theorem is left as an exercise for the student.

(5) *Zero has no real logarithm.*

In the equation $a^x = N$, suppose N is a very small positive number, then since $a > 1$, x must be a large negative number. As N approaches 0 the value of x remains negative and increases without limit. Thus no value of x can be found which will satisfy the equation $a^x = 0$.

(6) *A negative number has no real logarithm.*

Since a is positive, a^x is positive for all real values of x. Thus the equation $a^x = N$ is impossible if N is negative.

(7) *The logarithm of a product is equal to the sum of the logarithms of its factors.*

Let $M = a^x$, $N = a^y$, then $x = \log_a M$, $y = \log_a N$.
Now $MN = a^x \times a^y = a^{x+y}$. Hence

$$\log_a MN = x + y = \log_a M + \log_a N.$$

This formula may be extended to the product of any number of factors.

(8) *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.*

For $M/N = a^x/a^y = a^{x-y}$. Hence

$$\log_a M/N = x - y = \log_a M - \log_a N.$$

(9) *The logarithm of any power of a number is equal to the exponent multiplied by the logarithm of the number.*

For $M^k = (a^x)^k = a^{kx}$. Hence

$$\log_a M^k = kx = k \log_a M.$$

(10) *The logarithm of any root of a number is equal to the logarithm of the number divided by the index of the root.*

For $\sqrt[k]{M} = (a^x)^{1/k} = a^{x/k}$. Hence

$$\log_a \sqrt[k]{M} = x/k = \frac{1}{k} \log_a M.$$

NOTE. While (6) is true, negative numbers have logarithms which are complex numbers of the type $p + iq$. We shall not discuss such logarithms in this book.

EXERCISES

Verify the following:

1. $\log_2 32 = \log_2 8 + \log_2 4$.

2. $\log_3 27 = \log_3 243 - \log_3 9$.

3. $\log_2 256 = \log_2 16 + \log_2 8 + \log_2 2$.

4. $\log_5 3125 = \log_5 125 + \log_5 25$.

Find the value of

5. $\log_2 32^4$.

7. $\log_5 \sqrt[3]{625}$.

9. $\log_3 (1/27)^2$.

6. $\log_3 81^3$.

8. $\log_2 (\frac{1}{4})^5$.

10. $\log_4 \sqrt[5]{1024}$.

Given $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$, $\log_{10} 5 = 0.6990$; find the value of

11. $\log_{10} 6$.

13. $\log_{10} \frac{75}{2}$.

15. $\log_{10} \sqrt[3]{\frac{12}{5}}$.

12. $\log_{10} 120$.

14. $\log_{10} \frac{1}{50}$.

16. $\log_{10} \frac{\sqrt{15}}{\sqrt[3]{20}}$.

45. **Common Logarithms.** Any positive number (except 0 and 1) may be used as the base of a system of logarithms. However, in numerical calculations there are many advantages in using 10 as the base. Such logarithms are called ***common logarithms.*** In the remainder of this chapter we shall use common logarithms and consequently it will not be necessary to write the base. That is, we shall write $\log 36$ instead of $\log_{10} 36$.

The *natural logarithms* (base $e = 2.7182 +$) invented by NAPIER are very useful in theoretical calculations but are not used in elementary mathematics.

46. **Characteristic and Mantissa.** Since we are using common logarithms it is obvious that

$$\log 10 = 1, \qquad \log 0.1 = -1,$$
$$\log 100 = 2, \qquad \log 0.01 = -2,$$
$$\log 1000 = 3, \qquad \log 0.001 = -3.$$

Let us now suppose that the value of log 438 is desired; that is, suppose we wish to find the value of x which will satisfy the equation $10^x = 438$. Since 438 lies between 100 and 1000, it is evident that x lies between 2 and 3. Thus log 438 = 2 + (a decimal).

Likewise log 27.2 = 1 + (a decimal), log 6.47 = 0 + (a decimal), log 2534.5 = 3 + (a decimal), and so on. Thus the logarithm of every number greater than 1 is equal to some positive integer plus a decimal.

Let us now consider the logarithm of some number less than 1; for example, log 0.032. Since 0.032 lies between 0.01 and 0.1, its logarithm will lie between − 2 and − 1. Thus log 0.032 = − 2 + (a decimal).

Likewise log 0.4 = − 1 + (a decimal), log 0.005 = − 3 + (a decimal), log 0.0003 = − 4 + (a decimal), and so on. Thus the logarithm of every number between 0 and 1 is equal to a negative integer plus a decimal.

Since negative numbers have no real logarithms it is evident that every logarithm consists of an integral portion (positive or negative) which is called the ***characteristic,*** and a (positive) decimal portion which is called the ***mantissa.***

In practice, if the characteristic is negative, we add 10 to the logarithm and then subtract 10, which leaves its value unchanged but which results in a form better suited for computation. For example, if we find

$$\log 0.4 = -1 + 0.6021,$$

we write 9 + 0.6021 − 10 in place of − 1 + 0.6021. Then

$$\log 0.4 = 9.6021 - 10.$$

Likewise if we find log 0.005 = − 3 + 0.6990, we write log 0.005 = 7.6990 − 10.

It is evident that we cannot write log 0.4 = − 1.6021, since this means − 1 − 0.6021.

The theorems of § 44 lead to the discovery of several important facts with reference to the characteristic and the mantissa of a logarithm.

Suppose $\log 12.8 = 1.1072$, then

$$\log 128 = \log (10 \times 12.8) = \log 10 + \log 12.8$$
$$= 1 + 1.1072 = 2.1072,$$
$$\log 1280 = \log (100 \times 12.8) = 2 + 1.1072 = 3.1072,$$
$$\log 1.28 = \log \frac{12.8}{10} = \log 12.8 - \log 10$$
$$= 1.1072 - 1 = 0.1072,$$
$$\log 0.128 = \log \frac{12.8}{100} = 1.1072 - 2 = 9.1072 - 10,$$
$$\log 0.0128 = \log \frac{12.8}{1000} = 1.1072 - 3 = 8.1072 - 10.$$

The above results may now be summarized in the following statements:

(1) *If $N > 1$, the characteristic of $\log N$ is positive and is less by 1 than the number of digits to the left of the decimal point.*

(2) *If $0 < N < 1$, the characteristic of $\log N$ is negative and may be obtained as follows: Count the number of ciphers between the decimal point and the first significant digit, subtract this number from 9, write the remainder as a positive characteristic and annex -10 to the logarithm.**

Since negative numbers and 0 have no real logarithms, these two rules are sufficient for all numerical calculations. The characteristic is always determined in this way and is never found in a table of logarithms.

* If the number of ciphers before the first significant digit is greater than 9, subtract this number from 19 and annex -20, etc.

(3) *The mantissa of log N depends only on the order of the digits of the number N and is independent of the position of the decimal point.*

Since most logarithms are irrational numbers the mantissa is generally a non-repeating decimal, the true value of which may be only approximated. Tables have been constructed, however, from which this decimal may be found with a very high degree of accuracy. The four-place table on pages 104 and 105 is sufficiently accurate for most purposes.

EXERCISES

1. Write the characteristic of each of the following logarithms.

log 235.	log 54326.	log 0.041
log 562.37	log 0.3792	log 43.21
log 3.414	log 0.0043	log 0.00006

If log 2.43 = 0.3856, find the value of

2. log 243.	**4.** log 24.3	**6.** log 0.000243
3. log .0243	**5.** log 24300.	**7.** log 0.243

USE OF TABLE

47. **To Find the Logarithm of a Number.**

First, find the characteristic by one of the rules above.

Second, find the mantissa in the table in the following manner:

(1) *When the number consists of three digits.*

Find in the column *N* the first two digits of the number. Then the required mantissa will be found in the same horizontal line, in the column headed by the third digit of the number.

For example, log 21.6 = 1.3345, log 0.532 = 9.7259 − 10,
log 3.24 = 0.5105, log 0.00786 = 7.8954 − 10.

FOUR PLACE LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	16	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	4	5	7	9	10	12	14	16
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	12	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	5	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	7	8	9	11	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	2	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	4	5	6	7	8	9	11
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	6	7	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	5	6	7	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	3	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
N	0	1	2	3	4	5	6	7	8	9	1	2	2	4	5	6	7	8	9

FOUR PLACE LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1 2 2	3 4 5	5 6 7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1 2 2	3 4 5	5 6 7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1 1 2	3 4 5	5 6 7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1 1 2	3 4 4	5 6 7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1 1 2	3 4 4	5 6 7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1 1 2	3 4 4	5 6 6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1 1 2	3 3 4	5 6 6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1 1 2	3 3 4	5 5 6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1 1 2	3 3 4	5 5 6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1 1 2	3 3 4	5 5 6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1 1 2	3 3 4	5 5 6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1 1 2	3 3 4	5 5 6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1 1 2	3 3 4	5 5 6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1 1 2	3 3 4	4 5 6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1 1 2	3 3 4	4 5 6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1 1 2	3 3 4	4 5 6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1 1 2	3 3 4	4 5 6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1 1 2	3 3 4	4 5 6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1 1 2	2 3 4	4 5 5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1 1 2	2 3 4	4 5 5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1 1 2	2 3 3	4 5 5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1 1 2	2 3 3	4 4 5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1 1 2	2 3 3	4 4 5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1 1 2	2 3 3	4 4 5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1 1 2	2 3 3	4 4 5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1 1 2	2 3 3	4 4 5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1 1 2	2 3 3	4 4 5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1 1 2	2 3 3	4 4 5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1 1 2	2 3 3	4 4 5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1 1 2	2 3 3	4 4 5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1 1 2	2 3 3	4 4 5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1 1 2	2 3 3	4 4 5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	1 1 2	2 3 3	4 4 5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0 1 1	2 2 3	3 4 4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0 1 1	2 2 3	3 4 4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0 1 1	2 2 3	3 4 4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0 1 1	2 2 3	3 4 4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0 1 1	2 2 3	3 4 4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0 1 1	2 2 3	3 4 4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0 1 1	2 2 3	3 4 4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0 1 1	2 2 3	3 4 4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0 1 1	2 2 3	3 4 4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0 1 1	2 2 3	3 4 4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0 1 1	2 2 3	3 3 4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0 1 1	2 2 3	3 3 4
N	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9

(2) *When the number consists of less than three digits.*

Annex ciphers to the right and proceed as in (1).

For example, $\log 84 = \log 84.0 = 1.9243$,

$\log 0.7 = \log 0.700 = 9.8451 - 10.$

(3) *When the number consists of more than three digits.*

In this case we use the process called ***interpolation,*** which may best be explained by examples.

Example 1. Find the value of log 2457.

The given number lies between 2450 and 2460. We find from the table

the mantissa corresponding to	2450	is	3892,
" " " "	2460	"	3909,
differences	10		17.

Thus an increase of 10 in the number produces an increase of 17 in the mantissa. Then an increase of 7 in the number will produce an increase of 7/10 of $17 = .7 \times 17 = 11.9$ in the mantissa.

Hence

the mantissa corresponding to 2457 is $3892 + 11.9 = 3904$

to the nearest fourth decimal place.* Then $\log 2457 = 3.3904$

Note. The assumption here made that the increase in the number is proportional to the increase in the mantissa is not quite accurate, but when the increase in the number is small compared with the number itself, it is sufficiently accurate for practical purposes.

Example 2. Find the value of log 0.041532

The mantissa is independent of the position of the decimal point; hence the decimal point may be ignored. We find from the table

* When using four places of decimals the number in the fifth place is disposed of as follows. When it is less than 5 it is dropped; when it is more than 5, 1 is added to the number in the fourth place. When it is exactly 5, 1 is added to the number in the fourth place if that number is odd; otherwise it is dropped.

the mantissa corresponding to	41500	is	6180,
" " " "	41600	"	6191,
differences	100		11.

Hence the mantissa corresponding to 41532 is

$$6180 + \frac{32}{100} \text{ of } 11 = 6180 + .32 \times 11$$
$$= 6180 + 3.52 = 6184 \text{ to four places.}$$

Then $\log 0.041532 = 8.6184 - 10$.

The rule may now be stated as follows:

Find the mantissa corresponding to the first three digits. Find the tabular difference (17 and 11 above). Multiply the tabular difference by the remaining digits treated as a decimal and add the result to the first mantissa.

EXERCISES

Find the logarithm of each of the following numbers.

1. 24.	**5.** 937.81	**9.** 0.0454	**13.** 63.214
2. 32.41	**6.** 542.6	**10.** 1.0237	**14.** 0.00183
3. 4.786	**7.** 28.64	**11.** 16.463	**15.** 3456.2
4. 0.2143	**8.** 8.325	**12.** 0.0004	**16.** 5.2346

48. **To Find the Number Corresponding to a Given Logarithm.** The characteristic merely shows the position of the decimal point in the number required and is not to be found in the table. It should be disregarded until the sequence of digits corresponding to the given mantissa is found and the decimal point should then be inserted in accordance with (1) and (2) of § 46.

Thus, if the mantissa 8351 corresponds to the sequence 684,

the number whose logarithm is	1.8351	is	68.4
" " " "	3.8351	"	6840
" " " "	8.8351–10	"	0.0684

We find from the table the sequence of digits corresponding to the given mantissa in the following manner.

(1) *When the given mantissa is found in the table.*

In this case we can write the sequence of digits at once.

(2) *When the given mantissa is not found in the table.*

The method in this case is best shown by an example.

EXAMPLE. Find the number whose logarithm is 1.8737; that is, $\log x = 1.8737$, find x.

We find from the table

the mantissa	8733	corresponds to the sequence					747
" "	8739	"	"	"	"		748
differences	6						1

Also

the mantissa	8733	corresponds to the sequence					747
" "	8737	"	"	"	"		x
differences	4						?

Since an increase of 6 in the mantissa produces an increase of 1 in the number, an increase of 4 in the mantissa will produce an increase of 4/6 (= .67) in the number. Hence the sequence of digits corresponding to the mantissa 8737 is 74767. Then, since the characteristic of $\log x$ is 1, we must have two digits to the left of the decimal point, and $x = 74.767$

We have then the following rule:

Find the two successive mantissas between which the given mantissa lies, and note the three digits corresponding to the smaller mantissa. Subtract the smaller mantissa from the given mantissa, and divide the remainder by the tabular difference. Annex the quotient to the three digits already found and point off the result.

NOTE. The number which corresponds to a given logarithm is called its ***antilogarithm.*** Thus 100 is the antilogarithm of 2.

EXERCISES

Find the numbers whose logarithms are given below.

1. 1.4133	**6.** 0.1953	**11.** 9.2476 − 10.
2. 2.1346	**7.** 1.9985	**12.** 8.5342 − 10.
3. 0.7529	**8.** 3.7531	**13.** 6.3452 − 10.
4. 1.9542	**9.** 2.1478	**14.** 3.7513 − 10.
5. 2.2357	**10.** 4.5791	**15.** 5.9872 − 10.

49. **Computation with Logarithms.** The use of logarithms enables one to avoid much of the labor involved in the processes of multiplication, division, raising to powers, and extracting roots, as carried out in arithmetic.

We have already observed that, in general, a logarithm is an irrational number whose value can be only approximated. The table on pages 104 and 105 gives these values to four places of decimals. When using this table the results will not be accurate beyond four significant figures. In fact the fourth figure may be incorrect. If greater accuracy is desired, more extensive tables, such as *The Macmillan Tables,* should be used. Still more extensive tables exist and are used in computations that demand very great accuracy.

A negative number has no real logarithm. Consequently in computations involving such numbers we must first determine the sign of the result and then treat all negative numbers as if they were positive.

In the examples which follow we shall make frequent use of the theorems of § 44.

Example 1. Find the value of $.2478 \times 3.561 \times (-211.5)$.

Let N denote the product with the last factor positive. Then we may write

$$\log N = \log .2478 + \log 3.561 + \log 211.5,$$

and the calculation may be written in the form

$$\log .2478 = 9.3941 - 10$$
$$\log 3.561 = 0.5515$$
$$\log 211.5 = 2.3253$$
$$\log N = 12.2709 - 10$$
$$= 2.2709$$
$$N = 186.6$$

Hence the required value is -186.6

EXAMPLE 2. Find the value of $\sqrt[3]{\frac{12.64}{.0563}}$.

We may write

$$\log N = \tfrac{1}{3}(\log 12.64 - \log .0563),$$

or

$$3 \log N = \log 12.64 - \log .0563$$
$$\log 12.64 = 1.1018 \qquad = 11.1018 - 10$$
$$\log .0563 = 8.7505 - 10 = \underline{8.7505 - 10}$$
$$3 \log N = 2.3513$$
$$\log N = 0.7838$$
$$N = 6.078$$

NOTE. The solution of this example illustrates a device frequently employed when subtracting one logarithm from another.

EXAMPLE 3. Find the fifth root of .009276

Let $N = \sqrt[5]{.009276}$. Then $\log N = \frac{1}{5}\log .009276$

$$5 \log N = \log .009276$$
$$= 7.9674 - 10 = 47.9674 - 50.$$

Then

$$\log N = 9.5935 - 10$$
$$N = .3922$$

NOTE. Observe that 40 was added to and subtracted from the negative logarithm before dividing by 5. This device is frequently employed.

EXAMPLE 4. Find the value of $(27.56)^{3/4}$.

Let $N = (27.56)^{3/4}$. Then $\log N = \frac{3}{4} \log 27.56$

$$\log 27.56 = 1.4403$$
$$3 \log 27.56 = 4.3209$$
$$\log N = \tfrac{3}{4} \log 27.56 = 1.0802$$
$$N = 12.03$$

NOTE. The value of log 27.56 was *first* multiplied by 3 and *then* divided by 4. This reduces the error in the fourth place.

50. **Cologarithm.** Since $M/N = M(1/N)$, the logarithm of a fraction may be written as the sum, instead of the difference, of two logarithms. Thus

$$\log \frac{M}{N} = \log M + \log \frac{1}{N}.$$

The logarithm of the reciprocal of a number is called the ***cologarithm*** of the number. The cologarithm is very useful in avoiding alternate additions and subtractions. Since

$$\operatorname{colog} N = \log \frac{1}{N} = \log 1 - \log N = 0 - \log N$$
$$= -\log N = 10 - \log N - 10,$$

the value of the cologarithm may be quickly obtained by subtracting the logarithm from 10.0000 − 10.

EXAMPLE. Find the value of $\dfrac{21.6 \times 32.9}{3.35 \times 0.76}$.

$$\log N = \log 21.6 + \log 32.9 + \operatorname{colog} 3.35 + \operatorname{colog} 0.76$$
$$\log 21.6 = 1.3345$$
$$\log 32.9 = 1.5172$$
$$\operatorname{colog} 3.35 = 9.4750 - 10$$
$$\operatorname{colog} 0.76 = 0.1192$$
$$\log N = 12.4459 - 10 = 2.4459$$
$$N = 279.2$$

EXERCISES

Perform the following computations by logarithms.

1. 48×21.
2. 102×568.
3. $52 \times 39 \times 108$.
4. 78.42×0.000437
5. $27.6 \times 13.2 \times 68.3$
6. $6.42 \times 21.2 \times 0.12$
7. $88 \div 22$.
8. $123 \div 13$.
9. $564.2 \div 3.456$
10. $47.5 \div 0.058$
11. $0.0092 \div 768$.
12. 21.83^6
13. 0.035127^4
14. $(3.68 \times 0.97)^4$.
15. $\sqrt[4]{925}$.
16. $\sqrt[7]{-251}$.
17. $3\sqrt[4]{20}$.
18. $\sqrt[6]{1553}$.
19. $\sqrt[4]{.0359}$
20. $100^{3/2}$.
21. $714.2^{2/5}$
22. $(-3.41)^{2/3}$.
23. $2^{2/3} \times 15^{1/2}$.
24. $21 \times 3.6^4 \times .056^3$

Find the value of:

25. $\sqrt{42.6 \times 35.72 \times 0.562}$
26. $0.631 \times 0.7208 \times 0.51272$
27. $\dfrac{27.1 \times 324 \times 0.0034}{19.6 \times 41.2 \times 0.183}$.
28. $\sqrt[5]{\dfrac{126 \times (-234)}{62.5 \times (-165)}}$.
29. $9.345 \div (-0.0765)$.
30. $\sqrt[3]{5106.5 \times 0.3109}$
31. $\sqrt[3]{\dfrac{783 \times 8.31^2}{-24 \times 6.18^3}}$.
32. $\dfrac{\sqrt[3]{243}}{\sqrt{35}}$.

33. How many digits are there in 2^{20}?

34. The volume of a right circular cylinder is given by the formula $v = \pi r^2 h$. Find the volume if $r = 13.2$ inches, and $h = 40.5$ inches.

35. The volume of a sphere is given by the formula $v = \frac{1}{6}\pi d^3$. Find the volume if $d = 4.25$ inches.

36. The area of a triangle is expressed by the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where a, b, and c represent the lengths of the sides and

$$s = \tfrac{1}{2}(a + b + c).$$

What is the value of A when $a = 100$ feet, $b = 145.8$ feet, and $c = 126.2$ feet?

37. The crushing weight, P, in pounds, of an iron column is given by the formula

$$P = 299{,}600\,\frac{d^{3.55}}{l^2},$$

where d is the diameter in inches and l is the length in feet. What weight will crush an iron column 12.25 feet long, and 3.5 inches in diameter?

51. **Exponential Equations.** An equation in which the unknown appears in the exponent is called an ***exponential equation.*** Many exponential equations may be readily solved by logarithms.

EXAMPLE. Solve the equation $2^{x+3} = 5^{x-1}$.

We have

$$\log 2^{x+3} = \log 5^{x-1},$$

or

$$(x + 3)\log 2 = (x - 1)\log 5.$$

Then

$$x(\log 5 - \log 2) = \log 5 + 3\log 2,$$

or

$$x = \frac{\log 5 + 3\log 2}{\log 5 - \log 2} = \frac{0.6990 + 3(0.3010)}{0.6990 - 0.3010} = \frac{1.6020}{0.3980} = 4.03$$

EXERCISES

Solve each of the following equations.

1. $4^x = 8$.
2. $5^{2x+1} = 25$.
3. $5^x = 10$.
4. $12^x = .3$
5. $4^{2x+3} = 916$.
6. $7^{x+1} = 5$.
7. $3^{4x-1} = 4^{2x+3}$.
8. $10^{3x} = 324$.
9. $(4^x)^{x-2} = 64$.
10. The sum of the first n terms of a G. P. is

$$s = \frac{a - ar^n}{1 - r}.$$

Solve for n in terms of a, r, and s.

CHAPTER VI

MATHEMATICAL INDUCTION—BINOMIAL THEOREM

52. **Mathematical Induction.** In Chapter IV we derived certain formulas for the sum of the first n terms of an arithmetic progression, and of a geometric progression. We shall now derive these same results by another method which is applicable to a large number of mathematical formulas.

Let us consider the series 1, 2, 3, 4, $\cdots$, whose sum to n terms, as given by formula (2) of § 33, is

$$(1) \qquad 1 + 2 + 3 + 4 + \cdots + n = \frac{n}{2}(n + 1).$$

We propose now to establish this formula without making use of any of the results of Chapter IV.

Formula (1) is obviously true for certain small values of n. For example,

$$(n = 1) \qquad 1 = 1 = \frac{1}{2}(1 + 1),$$

$$(n = 2) \qquad 1 + 2 = 3 = \frac{2}{2}(2 + 1),$$

$$(n = 3) \qquad 1 + 2 + 3 = 6 = \frac{3}{2}(3 + 1),$$

$$(n = 4) \qquad 1 + 2 + 3 + 4 = 10 = \frac{4}{2}(4 + 1),$$

$$(n = 5) \qquad 1 + 2 + 3 + 4 + 5 = 15 = \frac{5}{2}(5 + 1).$$

From these results one might infer that formula (1) holds for all positive integral values of n. Such a conclusion, however, would be unjustified, since we have not *proved* the

formula but have merely *checked* it for certain special values of n.

One may easily set up a formula which is true for certain special values of n, but not for all values. Consider the formula

$$1 + 2 + 3 + 4 + \cdots + n = n^2 - 2n + 3.$$

When $n = 2$, we have $3 = 4 - 4 + 3 = 3$; and when $n = 3$, we have $6 = 9 - 6 + 3 = 6$. Since the formula holds when $n = 2$ and $n = 3$, we might infer that it holds for all positive integral values of n. But when $n = 4$ the formula gives $10 = 16 - 8 + 3 = 11$, and our inference is obviously incorrect.

Formula (1) may be established as follows:

Assume that the formula holds for some special value of n, say $n = r$. Then

$$(2) \qquad 1 + 2 + 3 + \cdots + r = \frac{r}{2}(r + 1).$$

Adding $r + 1$ to both members of this equation gives

$$\begin{aligned} 1 + 2 + 3 + \cdots + r + (r + 1) &= \frac{r}{2}(r + 1) + (r + 1) \\ &= (r + 1)\left(\frac{r}{2} + 1\right) \\ &= \frac{(r + 1)(r + 2)}{2}, \end{aligned}$$

which may be written in the form

$$(3) \qquad 1 + 2 + 3 + \cdots + (r + 1) = \frac{r + 1}{2}(r + 2).$$

But (3) is precisely what we would get if we should put $n = r + 1$ in (1). This shows that if formula (1) is true when n has any special value ($n = r$) it will also be true when the value of n is increased by 1 ($n = r + 1$).

We have seen that (1) is true when $n = 5$; hence it is true when $n = 6$. Since it holds for $n = 6$, it must hold for $n = 7$; etc. Hence (1) is true for all positive integral values of n.

The above method of reasoning is called ***mathematical induction.*** It should be observed that before we can infer that a mathematical formula holds for all positive integral values of n, we must show that

(1) the formula holds for some special value of n; and

(2) if the formula holds for a value of n, it will also hold when the value of n is increased by 1.

Other applications of this process will be found in the following examples.

EXAMPLE 1. Prove the correctness of the formula (§ 33)

$$a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d] = \frac{n}{2}[2a + (n - 1)d].$$

Assume the formula to be true. Adding $a + nd$ to both members of the equation gives

$$\begin{aligned} a + (a + d) + (a + 2d) + \cdots + (a + nd) &= \frac{n}{2}[2a + (n - 1)d] + a + nd \\ &= \frac{2an + (n^2 - n)d + 2a + 2nd}{2} \\ &= \frac{(n + 1)2a + (n^2 + n)d}{2} \\ &= \frac{n + 1}{2}(2a + nd). \end{aligned}$$

Now this is what we would get if we should replace n by $n + 1$ in the given formula. Hence if the formula holds for some small value of n it must hold for the next greater value of n. But the formula is obviously true when $n = 1$. Hence it holds for all positive integral values of n.

EXAMPLE 2. Prove the correctness of the formula (§ 37)

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a - ar^n}{1 - r}.$$

Assume the formula to be true and add ar^n to both members. We obtain

$$a + ar + ar^2 + \cdots + ar^n = \frac{a - ar^n}{1 - r} + ar^n$$

$$= \frac{a - ar^n + ar^n - ar^{n+1}}{1 - r} = \frac{a - ar^{n+1}}{1 - r},$$

which is the same as the given formula with n replaced by $n + 1$. Since the formula holds for $n = 1$, it must, therefore, be true for all positive integral values of n.

EXAMPLE 3. Prove the correctness of the formula

$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2.$$

Assume the formula to be true. Add to both sides of the equation the next odd number, $2n + 1$. We obtain

$$1 + 3 + 5 + 7 + \cdots + (2n + 1) = n^2 + 2n + 1 = (n + 1)^2,$$

which is the same as the given formula with n replaced by $n + 1$. But the given formula holds for $n = 1$. It must, therefore, be true for all positive integral values of n.

EXAMPLE 4. Prove the correctness of the formula

$$5 + 10 + 15 + 20 + \cdots + 5n = \frac{5n}{2}(n + 1).$$

Assume the formula to be true. Add to both sides of the equation the next multiple of 5. We obtain

$$5 + 10 + 15 + \cdots + (5n + 5) = \frac{5n}{2}(n + 1) + 5n + 5$$

$$= \frac{5(n + 1)}{2}(n + 2),$$

which is the same as the given formula with n replaced by $n + 1$. Hence the given formula must be true for all positive integral values of n, since it holds for $n = 1$.

EXERCISES

Prove the following statements by mathematical induction.

1. $2 + 4 + 6 + \cdots + 2n = n(n + 1)$.

2. $3 + 6 + 9 + \cdots + 3n = \frac{3n}{2}(n + 1)$.

3. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n}{6}(n + 1)(2n + 1)$.

4. $2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \frac{2n}{3}(n + 1)(2n + 1)$.

5. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2}{4}(n + 1)^2$.

6. $1^3 + 3^3 + 5^3 + \cdots + (2n - 1)^3 = n^2(2n^2 - 1)$.

7. $1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{n}{30}(n + 1)(2n + 1)(3n^2 + 3n - 1)$.

8. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \frac{n}{3}(n + 1)(n + 2)$.

9. $2 + 2^2 + 2^3 + \cdots + 2^n = 2(2^n - 1)$.

10. $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}$.

11. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}$.

12. $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n + 2) = \frac{n}{6}(n + 1)(2n + 7)$.

13. $2 \cdot 4 + 4 \cdot 6 + 6 \cdot 8 + \cdots + 2n(2n + 2) = \frac{n}{3}(2n + 2)(2n + 4)$.

14. The sum of the angles of a convex polygon is $(n - 2) \times 180°$.

15. The difference of the nth powers of two numbers is always divisible by the difference of the numbers. [HINT. Make use of the equation

$$a^{n+1} - b^{n+1} = a(a^n - b^n) + b^n(a - b)\text{]}.$$

53. **Proof of the Binomial Theorem.** In § 12 we stated the Binomial Theorem

$$(1)\qquad (a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots + nax^{n-1} + x^n,^*$$

and observed that it was true for a few special values of the exponent n. We shall now prove by mathematical induction that this theorem holds for *all* positive integral values of n.

Let us assume that the theorem is true for $n = k$, where k is some positive integer. Then

$$(2)\qquad (a+x)^k = a^k + ka^{k-1}x + \frac{k(k-1)}{2!}a^{k-2}x^2 + \frac{k(k-1)(k-2)}{3!}a^{k-3}x^3 + \cdots + kax^{k-1} + x^k.$$

Multiply both members of this equation by $a + x$ and combine the coefficients of like powers of x. We obtain the formula

$$(3)\qquad (a+x)^{k+1} = a^{k+1} + (k+1)a^k x + \frac{(k+1)k}{2!}a^{k-1}x^2 + \frac{(k+1)k(k-1)}{3!}a^{k-2}x^3 + \cdots + (k+1)ax^k + x^{k+1}.$$

But (3) is precisely (2) with $k + 1$ substituted for k throughout. Hence if (1) holds for some special value of n, it will hold for the next greater value. But we have seen (§ 12) that it holds for certain small positive integral exponents. Hence the Binomial Theorem holds for all positive integral exponents.

* The symbol $r!$, read "factorial r," means the product of the first r integers, $1 \cdot 2 \cdot 3 \cdots r$.

54. **The Binomial Theorem for Any Real Power.** It can be shown that the Binomial Theorem holds, not only for positive integral values of the exponent, but also for any other real value, *provided that x is numerically less than a.* The proof of this fact is beyond the scope of this book and will be omitted. It should be observed that in this case the left member, instead of coming to an end, will contain infinitely many terms. Why?

EXAMPLE. Expand the expression $(a - x)^{1/2}$ to four terms.

$$(a - x)^{1/2} = a^{1/2} + \tfrac{1}{2}a^{(1/2)-1}(-x) + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2!}a^{(1/2)-2}(-x)^2 + \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)}{3!}a^{(1/2)-3}(-x)^3 + \cdots$$

$$= a^{1/2} - \tfrac{1}{2}a^{-1/2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{1\cdot 2}a^{-3/2}x^2 - \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1\cdot 2\cdot 3}a^{-5/2}x^3 + \cdots$$

$$= a^{1/2} - \tfrac{1}{2}a^{-1/2}x - \tfrac{1}{8}a^{-3/2}x^2 - \tfrac{1}{16}a^{-5/2}x^3 + \cdots.$$

It is obvious that this expansion may be extended to any desired number of terms.

55. **Roots of Numbers.** If the value of x in $(a + x)^n$ is small in comparison with that of a, the binomial formula furnishes a very convenient method for finding approximate values of the roots of numbers.

EXAMPLE 1. Find the value of $(1.06)^5$.

$$(1.06)^5 = (1 + .06)^5$$

$$= 1 + 5(.06) + \frac{5\cdot 4}{1\cdot 2}(.06)^2 + \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}(.06)^3 + \cdots$$

$$= 1 + .3 + .036 + .00216 + \cdots = 1.338\cdots.$$

By writing the two terms omitted above, the student may show that they would not change the answer in the third decimal place.

EXAMPLE 2. Find the value of $\sqrt[3]{7.56}$

Write the given number as the sum, or difference, of two numbers, the first of which is a perfect cube.

$$\sqrt[3]{7.56} = (8 - 0.44)^{1/3} = 8^{1/3} + \frac{1}{3}8^{(1/3)-1}(-0.44) + \frac{\frac{1}{3}(-\frac{2}{3})}{2!}8^{(1/3)-2}(-0.44)^2 + \cdots$$

$$= 8^{1/3} - \frac{1}{3}8^{-2/3}(0.44) - \frac{\frac{1}{3}(\frac{2}{3})}{1\cdot 2}8^{-5/3}(0.44)^2 + \cdots$$

$$= 2 - \frac{1}{3\cdot 4}(0.44) - \frac{1}{9\cdot 32}(0.44)^2 + \cdots.$$

The value of each term after the second is very small and may be neglected. Then $\sqrt[3]{7.56} = 2 - 0.037 = 1.963$, approximately. This value agrees to two places of decimals with that given in the tables.

EXERCISES

Making use of the binomial formula, find an approximate value of each of the following:

1. $\sqrt{10}$.
2. $\sqrt[3]{26}$.
3. $\sqrt{123}$.
4. $\sqrt[3]{1.32}$
5. $\sqrt[3]{219}$.
6. $\sqrt[4]{78}$.
7. $\sqrt[3]{724}$.
8. $\sqrt[5]{33}$.

Expand to four terms:

9. $(1 + x)^{1/3}$.
10. $\sqrt[5]{a - x}$.
11. $(x - y^2)^{-4}$.
12. $(1 + 2x)^{-4}$.
13. $(2a - 5b)^{-6}$.
14. $\left(\frac{a}{b} - \frac{b}{a}\right)^{-5}$.

Making use of the formula of § 13 find the

15. 4th term of $(a - x)^{2/3}$.
16. 3rd term of $(2x + 3y)^{-4}$.
17. 7th term of $(1 - x)^{-1}$.
18. 6th term of $(1 + a^2b^{-1/3})^{-3}$.
19. 6th term of $(a^{5/2} - 4b^{-2})^{-3/4}$.
20. 5th term of $(c^{-2} + e^{-1/2})^{-4}$.

21. Expand $(1 - x)^{-1}$ to five terms by the binomial formula. Show that the expansion is valid when $x = \frac{1}{2}$, but not when $x = 2$. Why?

CHAPTER VII

GRAPHICAL REPRESENTATION OF NUMBERS

56. **Vectors.** If A and B are two points on an indefinite line $X'X$, the segment of the line terminated by the points

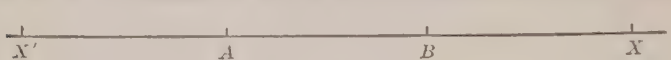


FIG. 3

A and B may be thought of as traced by a point moving along $X'X$ from A to B or from B to A. In the former case the line-segment is said to have the direction from A to B, and is called the ***vector AB;*** in the latter case it is said to have the direction from B to A, and is called the ***vector BA.*** Thus a vector represents a certain *length* in a certain *direction.* A vector is sometimes called a ***directed line-segment.***

The notation indicates the direction of the vector in each case. Thus the vector AB is thought of as beginning at a certain definite point A and ending at another definite point B. If two vectors have equal lengths and the same direction, they are said to be equal.

57. **Graphical Representation of Real Numbers.** Let $X'X$ be an indefinite straight line (Fig. 4), and let a fixed

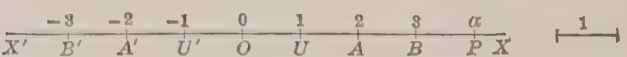


FIG. 4

point O, called the ***origin,*** be chosen upon it. Let U be another fixed point on the line one unit to the right of O, and suppose a number of segments of unit length are laid

off on the line to the right and left of O. Then, if we consider all vectors measured toward the right as positive and those measured toward the left as negative, the vectors OU, OA, OB, OU', OA', OB' may be denoted by the numbers 1, 2, 3, -1, -2, -3, respectively. That is,

$$\begin{aligned} OU &= 1, & OA &= 2, & OB &= 3, \\ OU' &= -1, & OA' &= -2, & OB' &= -3. \end{aligned}$$

In general, if P is any point on the line such that $OP = (p/q)\cdot OU$, where p and q are integers, the vector OP may be said to represent the rational number p/q. Conversely, if a rational number p/q be given, there will exist one and only one point P on the line such that the vector OP will represent the number p/q, both in magnitude and sign. The point P will be to the right or left of O according as the number p/q is positive or negative.

Let us now see whether an irrational number such as $\sqrt{2}$ may be represented by a vector. It is always possible to find a rational number whose value approximates a given irrational number with any degree of accuracy. Thus each of the following sequences of numbers

$$1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ \cdots,$$

and

$$2,\ 1.5,\ 1.42,\ 1.415,\ 1.4143,\ \cdots,$$

may be made to approximate the value of $\sqrt{2}$ as near as one pleases by increasing the number of terms indefinitely. Let

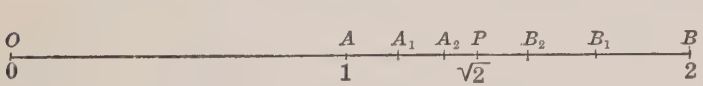


FIG. 5

OA, OA_1, OA_2, $\cdots$, OA_r, $\cdots$ represent the rational numbers of the first sequence, and OB, OB_1, OB_2, $\cdots$, OB_r, $\cdots$

represent the rational numbers of the second sequence. The distance between A_r and B_r may be made as small as one pleases by taking r sufficiently large. That is, the segments OA_r and OB_r approach the same segment OP when r is indefinitely increased. This vector OP will, therefore, represent the irrational number $\sqrt{2}$.

In a similar manner it may be shown that a vector OP exists which will represent any given irrational number.

From what precedes we see that, if any *real* number a be given, there exists one and only one point P on the line, such that the vector OP will represent the number a both in magnitude and sign; and, conversely, if a point P be given, there exists one and only one real number a which will be represented by the vector OP, both in length and direction.

58. **Graphical Addition of Real Numbers.** We have seen that the result of any of the operations of addition, subtraction, multiplication, and division, on real numbers is a real number. Since every real number may be represented by a vector, it is possible to perform these operations graphically. We shall consider only addition.

Let us suppose that the real numbers a and b are represented by the vectors AB and BC. We now propose to find a vector which will represent the real number $a + b$.

FIG. 6

If both a and b are positive, the line segments may be added as in the first figure and it is evident that $AB + BC = AC$. If a is positive and b negative, as in the second figure, $AB + BC = AB - CB = AC$.

The rule for the addition of two vectors representing real numbers may now be stated as follows:

Lay off the vector OP to represent the number a. Beginning at P, the terminal point of the first vector, lay off the vector PQ to represent the number b. Then the vector OQ will represent the number a + b. This is a special case of § 61.

59. **Graphical Representation of Pure Imaginary Numbers.** We have seen that any positive real number a may be represented by the vector OA and the real number $-a$ by the vector OA', having the same length as OA but extending in the opposite direction.

Since $-a = (-1) \times (+a)$, we may think of -1 as an operator which turns the vector which represents the number a through two right angles in the counter-clockwise direction.

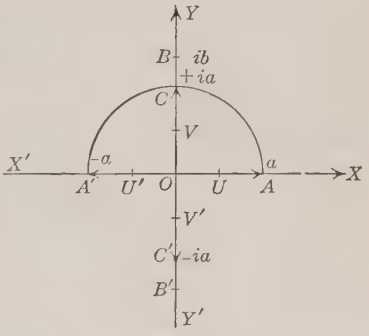


Fig. 7

Now $-1 = i \times i$ (§ 16). That is, $-a = i \times i \times (+a)$. Thus one multiplication by i, followed by a second multiplication by i, turns the vector which represents the number a through two right angles. We may, therefore, regard multiplication of a number by i as turning the vector which represents the number through one right angle in the counter-clockwise direction. Evidently we may write

$$ia = i \times (+a);$$

hence the vector OC represents the number ia.

$$-a = i \times i \times (+a);$$

hence the vector OA' represents the number $-a$. Moreover

$$-ia = i \times i \times i \times (+a);$$

hence the vector OC' represents the number $-ia$.

The geometric meaning of the statement on page 48, at the end of § 16, with reference to integral powers of i is now evident.

Thus if $X'X$ and $Y'Y$ be two directed lines intersecting at O at right angles, every real number a may be represented by a vector OA of length a measured along $X'X$ and every pure imaginary number ib by a vector OB of length b measured along $Y'Y$. In particular, the pure imaginary number i is represented by a vector OV of unit length measured from O toward Y.

60. Graphical Representation of Complex Numbers. Since the complex number $a + ib$ is the sum of the real number a and the pure imaginary number ib, it may be represented by the vector which is the sum of the vectors OA and OB.

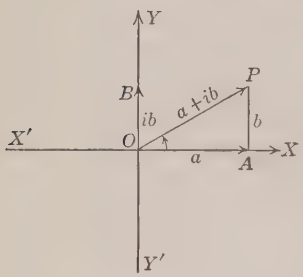


FIG. 8

In order to obtain the sum of these vectors we proceed as in § 58. Beginning at A, the terminal point of the vector OA, lay off the vector AP equal (and consequently parallel) to OB. The vector OP will represent the complex number $a + ib$.

The figure represents the case where a and b are both positive. The method is the same in any case.

The ***absolute value,*** or *modulus,* of a real, pure imaginary, or complex number is the *length* of the vector which represents the number. The modulus of a number is usually taken positive.

Thus the modulus of $3 + 4i$ is $\sqrt{3^2 + 4^2} = 5$; and the modulus of $a + ib$ is $\sqrt{a^2 + b^2}$.

The angle through which the line OX must be rotated in the counter-clockwise direction in order to take the direction of the vector OP is called the ***angle*** of OP, or the *argument* of OP. It is evident that the number of degrees in the angle of the number $a + ib$ depends upon the ratio b/a of the real parts of the number, since the angle is the same if b/a is the same, but changes when b/a changes.

Thus the angle of a positive real number is 0°; of a negative real number, 180°; of a positive pure imaginary number, 90°; of a negative pure imaginary number, 270°.

We may summarize the above results in the following statement. If a complex number $a + ib$ be given, there will exist one and only one vector OP drawn from the origin in the plane XY which will represent the number $a + ib$, both in absolute value and angle; and, conversely, if a vector OP be given, there will exist one and only one complex number $a + ib$, which will represent the vector OP, both in length and in direction. This representation of complex numbers by vectors is therefore perfect.

61. **Graphical Addition of Complex Numbers.** The foregoing method of adding graphically a real and a pure imaginary number is a general one and may be used for the addition of any two complex numbers.

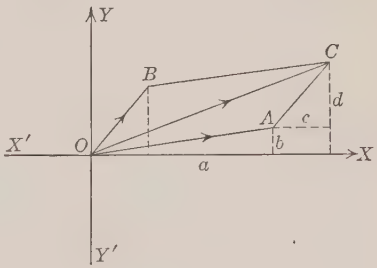


FIG. 9

Let $a + ib$ be represented by the vector OA and $c + id$ by the vector OB. From A draw the vector AC equal to the vector OB. The vector OC will then represent the sum of the vectors OA and OB.

This method of vector addition may be described as follows:

Complete the parallelogram of which OA and OB are adjacent sides. Then the diagonal OC will represent the sum of the vectors OA and OB.

Any two directed magnitudes, such as forces or velocities in physics, may be added in this way.

The result of each of the operations of addition, subtraction, multiplication, division, raising to powers, and extracting roots, upon complex numbers may be exhibited graphically by means of vectors although they will not be considered here.

EXERCISES

Draw a figure showing the vectors which represent the following numbers. Find their absolute values and indicate their angles.

1. 4.	**5.** $\sqrt{3}$.	**9.** $\sqrt{-9}$.	**13.** $12 + 5\sqrt{-1}$.
2. -5.	**6.** $-\sqrt{5}$.	**10.** $-\sqrt{-16}$.	**14.** $12 - 5\sqrt{-1}$.
3. $2\frac{1}{2}$.	**7.** π.	**11.** $\sqrt{-2}$.	**15.** $-12 + 5\sqrt{-1}$.
4. $-7\frac{1}{3}$.	**8.** $-\pi$.	**12.** $-\sqrt{-3}$.	**16.** $-12 - 5\sqrt{-1}$.

17. Show that the vector OC (Fig. 9) represents the number $(a + c) + i(b + d)$.

18. Draw the vectors which represent the complex numbers $4 + i$ and $2 + 3i$. Add these numbers graphically.

19. The two numbers $a + ib$ and $a - ib$ are called *conjugate* complex numbers. Show graphically that their sum is a real number. Is their product a real number?

20. Show that the sum of the angles of two conjugate complex numbers is 360°.

21. Show that the absolute value of the sum of two positive real numbers is equal to the sum of the absolute values. Is this true in any other case?

22. Two forces are acting on a body. The first, 120 pounds, is acting in the direction 60° north of west. The second, 50 pounds, is acting in the direction 30° south of west. Draw a figure showing the magnitude and direction of the equivalent single force.

23. A boat is moving south at the rate of 6 miles per hour. A man walks across the deck directly east at 6 miles per hour. How rapidly is the man moving with reference to the earth and in what direction?

62. **Graphical Representation of a Pair of Real Numbers.** Let $X'X$ and $Y'Y$ be two directed lines intersecting at O at right angles. Suppose any two real (positive or negative) numbers x and y are given. Measure off from O along $X'X$ a vector OM corresponding to the real number x and along $Y'Y$ a vector ON corresponding to the real number y. Complete the rectangle $MONP$. Then (§ 60) the vector OP corresponds to the single complex number $x + iy$.

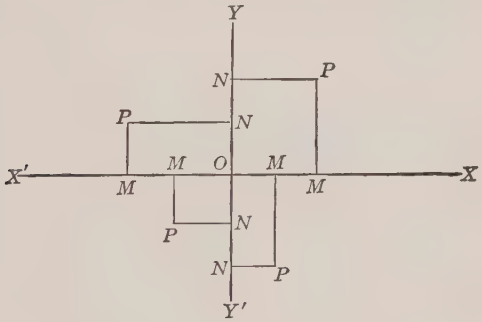


FIG. 19

Since the position of the point P in the plane is thus uniquely determined by the pair of real numbers x and y, we may consider this point as the graphical representation of the given pair of numbers. To each pair of real numbers x and y there corresponds a single point P in the plane of the lines, and to each point P in the plane there corresponds a single pair of real numbers x and y.

The choice of the positive direction on each of the axes is entirely arbitrary. However, it is customary to consider OX (to the right) as the positive direction on $X'X$ and OY (upward) as the positive direction on $Y'Y$.

The vector OM $(= NP)$ is called the ***abscissa,*** and the vector ON $(= MP)$ is called the ***ordinate,*** of the point P. The abscissa and ordinate together are called the ***rectangular coordinates*** of the point P.

The lines $X'X$ and $Y'Y$ are called the ***coordinate axes.*** Their point of intersection is called the ***origin.*** $X'X$ is also called the *x axis* or the *axis of abscissas,* and $Y'Y$ the *y axis* or the *axis of ordinates.*

It is customary to denote the numbers represented by the vectors OM and ON by x and y, respectively. The point P is then spoken of as the point whose coordinates are x and y, and it is denoted by $P(x, y)$ or more simply by (x, y). It is evident that each of these numbers may be either positive or negative.

Care should be taken not to confuse the graphical representation of a pair of real numbers by a point with the representation of a single complex (or real) number by a vector. The *points* M, N, and P represent the *pairs* of real numbers $(x, 0)$, $(0, y)$, and (x, y), respectively. The *vectors* OM, ON, and OP represent the *single* complex (or real) number x, iy, and $x + iy$, respectively.

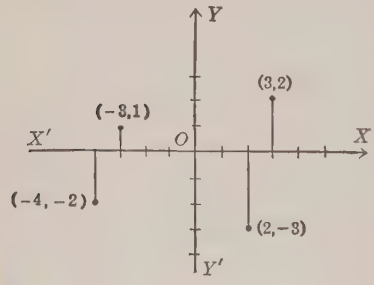


FIG. 11

The method of locating, or plotting, points, illustrated in Figure 11, is as follows:

Measure off from O along the x axis a vector equal to the given abscissa, then from the end-point of this vector measure

off along a line parallel to the y axis a vector equal to the given ordinate. The end-point of this vector will be the desired point.

EXERCISES

1. Plot the following pairs of numbers:

(*a*) (2, 3), (3, 5), (− 2, 4), (− 3, − 3), (5, − 4).

(*b*) (5/2, 2), (− 3,2, 7/2), (3, − 3/2), (− 4, − 13/2), (6, − 2/3).

2. Plot the following pairs of numbers:

(*a*) (2, 4), (2, − 2), (2, 0), (2, − 4), (2, 3/2).

(*b*) (2, 5), (− 3, 5), (0, 5), (− 5/2, 5), (4, 5).

3. Describe the position of all points which have the

(*a*) abscissa 3. (*c*) abscissa − 2.

(*b*) ordinate 4. (*d*) ordinate − 5.

4. Interpret geometrically each of the following equations:

(*a*) $x = 4$. (*b*) $y = 2$. (*c*) $x = -5/2$. (*d*) $y = -4$.

5. If a point moves parallel to the x axis, which of its coordinates remains constant?

6. If the abscissa of a point is zero, what may be said about the position of the point?

7. Where do all points lie whose abscissas and ordinates are equal?

8. Where do all points lie whose abscissas and ordinates have the same numerical value but opposite signs?

9. Interpret geometrically each of the following equations:

(*a*) $x = y$. (*b*) $x = -y$.

CHAPTER VIII

LINEAR FUNCTIONS

63. **Variables and Constants.** In the preceding chapters the algebraic symbols have in every case represented quantities whose values were fixed, although perhaps unknown. Such quantities are called ***constants.*** Thus in the equation $x^2 - 3x + 18 = 0$ the symbol x represents either of the fixed numbers which satisfy the equation.

A quantity which is free to take every possible value and to change from any one value to any other during a given discussion is called a ***variable.*** Most of the quantities of nature such as time, temperature, etc., are variables.

64. **Variation.** When any two variables are so related that their ratio never changes, though the variables themselves may change, either variable is said to *vary directly as the other.*

Thus, if an automobile uses a certain amount of gasoline per hour, the total amount used per day varies directly as the number of hours the car is driven.

65. **Inverse Variation.** If a variable y varies as the reciprocal of a variable x, then y is said to *vary inversely as* x.

Thus, the time required to walk a certain distance varies inversely as the rate of walking, for if the rate is doubled the time is halved, etc.

66. **Joint Variation.** If a variable y varies directly as the product of two other variables, x and z, it is said to *vary jointly* as x and z.

Thus, the interest received in a year from an investment varies jointly as the principal and the rate.

67. **Equations Expressing Types of Variation.** Suppose x, y, and z are variables and m is some constant. Then the above definitions may be stated as follows:

(1) *y varies directly as x when $y/x = m$, or $y = mx$.*

(2) *y varies inversely as x when $y = m(1/x)$, or $xy = m$.*

(3) *y varies jointly as x and z when $y/(xz) = m$, or $y = mxz$.*

Example. The surface of a cube varies as the square of its edge. If the surface of a cube whose edge is 10 inches is 600 square inches, what will be the edge of a cube whose surface is $30\frac{3}{8}$ square inches?

First Solution. Let e_1, e_2, s_1, s_2 represent the length of the edge and the area of the surface under the two conditions. Then

$$\frac{s_1}{s_2} = \frac{e_1^2}{e_2^2} = \left(\frac{e_1}{e_2}\right)^2;$$

whence

$$\frac{600}{30\frac{3}{8}} = \left(\frac{10}{e_2}\right)^2, \qquad \text{or} \qquad \left(\frac{10}{e_2}\right)^2 = \frac{4800}{243} = \frac{1600}{81},$$

and

$$\frac{10}{e_2} = \frac{40}{9}. \quad \text{Then } e_2 = \frac{90}{40} = 2\tfrac{1}{4} \text{ inches.}$$

Second Solution. Let $s = me^2$, then

$$600 = m(10)^2 = 100m.$$

Hence $m = 6$ and the relation between s and e becomes

$$s = 6e^2.$$

Now put $30\frac{3}{8}$ for s and we find $e = 2\frac{1}{4}$ inches.

EXERCISES

State the following laws of variation. Convert each statement into an equation containing the proper value of m (§ 67).

1. Perimeter of a square with its edge.

2. Circumference of a circle with its diameter.

3. Area of a circle with its radius.

4. Area of a triangle with its height and base.

5. Volume of a cylinder with its height and radius.

6. Volume of a sphere with its diameter.

7. If 5 men can do a piece of work in 40 days, how long will it take 20 men to do it?

8. The edges of two cubes are 3 feet and 4 feet respectively. Find the ratio of their volumes.

9. If y varies inversely as x and $y = 32$ when $x = 6$, find the value of y when $x = 21$.

10. If y varies jointly as x^2 and z^3, and $y = 6$ when $x = 4$ and $z = \frac{1}{2}$, what is the relation between x, y, and z?

11. If the diameter of a circle be doubled, how will its area be affected?

12. If y varies as x^3 and is equal to 40 when $x = 4$, what is the relation between y and x?

13. If y varies as x and x varies as z, show that y varies as z.

14. If y varies as z^2 and z varies inversely as x, show that xy varies inversely as x.

15. The number of posts required for a fence varies inversely as the distance between them. If it takes 90 posts when they are placed 12 feet apart, how many will be required when they are placed 15 feet apart?

16. If the diameter of a sphere be increased by 10%, by what percent will the volume be increased?

17. If y equals the sum of three numbers, the first of which is constant, and the second and third vary as x and x^2, respectively, and $y = 4$ when $x = 1$, 12 when $x = 2$, 26 when $x = 3$, find the value of y when $x = -1$.

18. The number of feet a body falls varies as the square of the number of seconds occupied in falling. If the body falls 16.1 feet the first second, how many feet will it fall in 4 seconds?

19. The squares of the periods of revolution of the planets about the Sun are proportional to the cubes of their distances from the Sun. If it takes Mars 687 days to go around the Sun, what is the relation between its distance, and the Earth's distance, from the Sun?

20. The pressure of wind on a plane surface varies jointly as the area of the surface and the square of the wind's velocity. The pressure on a square foot is 1 pound when the wind is moving at the rate of 15 miles per hour; find the velocity of the wind when the pressure on a square yard is 16 pounds.

68. **Functions.** When one variable y so depends on another variable x that a given value of x determines one or more values of y, then y is said to be a ***function*** of x.

Many examples of functions are to be found in nature. Thus the number of bushels of wheat which may be grown on a given acre of land is a function of (depends on) the character of the soil, time of planting, climatic conditions, and many other variable quantities. The relation between these variables cannot as yet be expressed in mathematical terms. In fact man, in his conquest of nature, is continually searching for some mathematical equation which will exactly express the relation between the variable quantities which he encounters.

The following statements are examples of certain functions where this relation has been found.

(1) The length of the circumference of a circle depends upon the length of its diameter. Thus $c = \pi d$.

(2) The area of a square depends upon the length of its side. Thus $A = s^2$.

(3) The time it takes to go a certain distance depends upon the velocity. Thus $t = d/v$.

The student should state the type of variation defined by each of the above equations.

When we wish to state that y is a function of x without specifying the relation between the variables we may write $y = f(x)$. We read $f(x)$ as "f of x." After the symbol $f(\ )$ has been defined, it must denote the same operation throughout the discussion. Thus, if

$$f(x) = x^2 - 3x + 4,$$

then

$$f(a) = a^2 - 3a + 4, \qquad f(1) = 1 - 3 + 4 = 2,$$
$$f(3) = 9 - 9 + 4 = 4, \qquad f(-2) = 4 + 6 + 4 = 14.$$

EXERCISES

1. Given $f(x) = 2x - 4$, find $f(1)$, $f(2)$, $f(-2)$, $f(3)$, $f(0)$.

2. Given $f(x) = x^2 + 4x - 12$, find $f(1)$, $f(2)$, $f(-3)$, $f(5)$, $f(-6)$.

3. Express the area of a circle as a function of its diameter.

4. The hypotenuse of a right triangle is 10. Express one of the perpendicular sides as a function of the other.

5. Given $f(x) = \sqrt{16 - x^2}$, find $f(0)$, $f(2)$, $f(3)$, $f(4)$.

6. Given $f(x) = (x^2 - 1)^2$, show that

$$f(x + 1) - f(x - 1) = 8x^3.$$

69. **The Graph of a Function.** Suppose $f(x) = 2x$. For every assigned value of x there is a corresponding value of $f(x)$. Several of these pairs of numbers are shown in the following table.

x	0	1	2	3	4	5	-1	-2	-3	-4
$f(x)$	0	2	4	6	8	10	-2	-4	-6	-8

Let us now draw a set of rectangular axes and plot these pairs of numbers (§ 62), taking the values of x as abscissas and the corresponding values of $f(x)$ as ordinates. The graph of each of these pairs of numbers will be a point (Fig. 12) and the points thus obtained appear to lie on a straight line.

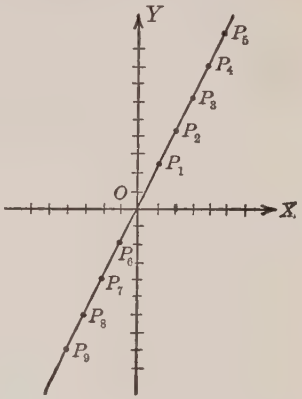


FIG. 12

In general, if any function of x, say $f(x)$, be denoted by y, it is possible to find infinitely many pairs of numbers which will satisfy the equation $y = f(x)$. To each pair of numbers there corresponds a definite point. Each of these points will lie on a certain curve which is called the ***graph of the function.***

In order to obtain an approximation to the graph of a function it is necessary to plot a number of points, whose coordinates satisfy the equation, and then to draw a smooth curve through these points.

70. **The Zeros of a Function.** If a function of x reduces to zero for certain values of x, these numbers are said to be the ***zeros*** of the function.

Thus, for each of the numbers 2 and 5 the function of x defined by $f(x) = x^2 - 7x + 10$ becomes equal to zero. These two numbers are, therefore, the zeros of the function.

It is obvious that if a is one of the zeros of $f(x)$, that is, if $f(a) = 0$, the curve which is the graph of $f(x)$ will either cross or touch the x axis at the point whose abscissa is a; and conversely, if the graph of a function $f(x)$ crosses or touches the x axis at a point whose abscissa is a, then $f(a) = 0$ and the number a is one of the zeros of $f(x)$.

This suggests a method for finding an approximation to the roots of an equation $f(x) = 0$, which is called ***solving an equation graphically.***

Set $y = f(x)$. Draw the graph of the resulting equation and estimate the abscissas of the points where the curve crosses the x axis.

71. The Graph of a Linear Function. The most general expression of the first degree in the variable x may be written in the form

$$f(x) = mx + b,$$

where m and b are constants. We shall consider only the cases where these constants are real numbers.

We have seen that in the special case where $m = 2$ and $b = 0$, the graph of this function seemed to be a straight line. We shall now show that *the graph of every function of the first degree is a straight line.*

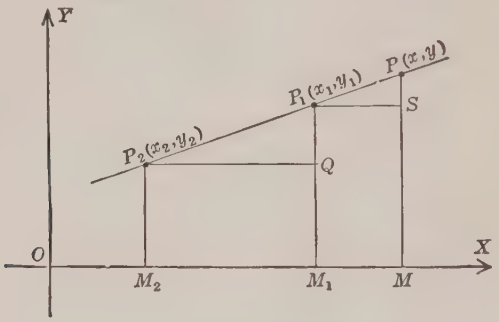


FIG. 13

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two fixed points on the graph of the function $f(x) = mx + b$, and $P(x, y)$ any variable point on the graph. The coordinates of each of these points must satisfy the equation

$$y = mx + b. \tag{1}$$

Hence

$$y_1 = mx_1 + b, \tag{2}$$

and

$$y_2 = mx_2 + b. \tag{3}$$

Subtracting (2) from (1) gives

$$y - y_1 = m(x - x_1),$$

and subtracting (3) from (2) gives

$$y_1 - y_2 = m(x_1 - x_2).$$

Hence by division,

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}, \quad \text{or} \quad \frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2},$$

or

$$\frac{SP}{P_1S} = \frac{QP_1}{P_2Q},$$

which can be true only when P_2P_1P is a straight line.

Since $P(x, y)$ may represent any point on the graph, we have shown that every point on the graph lies on the straight line determined by the two points P_1 and P_2. Hence the graph of every function of the first degree is a straight line. Such a function is, therefore, called a ***linear function.***

From equations (2) and (3) we obtain, by subtraction, $y_1 - y_2 = m(x_1 - x_2)$, or

$$m = \frac{y_1 - y_2}{x_1 - x_2}.$$

But (Fig. 14)

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{M_1P_1 - M_2P_2}{OM_1 - OM_2} = \frac{QP_1}{P_2Q}.$$

As a point moves along the line $y = mx + b$ from P_2 to

P_1 it will rise a distance QP_1, while moving through the horizontal distance P_2Q. The constant ratio, QP_1/P_2Q, which is always equal to m, is called the ***slope*** of the line.

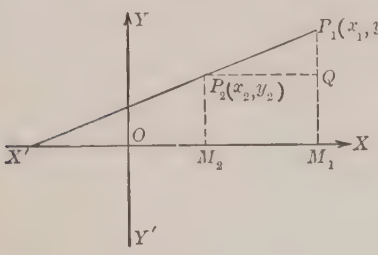


FIG. 14

The slope m is the ratio of the change in the ordinate to the change in the abscissa as a point moves along the line. It measures the steepness of the line, that is, the rate at which the line rises or falls. It is positive if both the abscissa and ordinate increase or decrease as the point moves. It is negative if one increases while the other decreases. In particular, the slope of a line parallel to the x axis is zero.

Suppose $m \neq 0$. Then $f(x) = mx + b = 0$ only when $x = -b/m$. Hence *a linear function has only one zero* (§ 70); that is, the line crosses the axis only once. If $m = 0$, then $f(x) = b$ and the line is parallel to the x axis.

If $B \neq 0$, the most general linear equation in x and y,

$$Ax + By + C = 0,$$

may be written in the form

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

This is in the form $y = mx + b$, where

$$m = -\frac{A}{B}, \qquad b = -\frac{C}{B}.$$

That is, the general linear equation $Ax + By + C = 0$ has a straight line for its graph if $B \neq 0$.

If $B = 0$, the original equation reduces to $Ax + C = 0$, or $x = -C/A$, which represents a straight line parallel to the y axis. Hence *the graph of every linear equation in two variables is a straight line.*

EXAMPLE. Draw the graph of the equation

$$3x - 2y - 6 = 0.$$

Since the graph is a straight line, it is necessary to plot only two points in order to draw the graph.

If $x = 0$, then $y = -3$; if $y = 0$, then $x = 2$. That is, the line cuts the x and y axes in the points $(2, 0)$ and $(0, -3)$, respectively. The line through these two points will be the graph of the given equation.

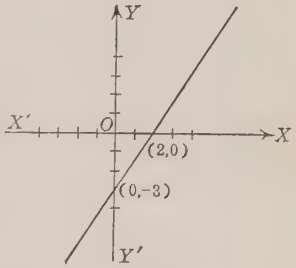


FIG. 15

EXERCISES

Draw the graphs of the following equations. Find their slopes and the coordinates of the points where they cut the axes.

1. $y = 3x$.
2. $y = -2x$.
3. $y = 2x + 4$.
4. $y = 2x - 5$.
5. $y = 6$.
6. $2y + 3 = 0$.
7. $y = -4x + 3$.
8. $y = -4x - 8$.
9. $x + y = 6$.
10. $x - y = 8$.
11. $x = 5$.
12. $3x - 7 = 0$.
13. $4x - y = 0$.
14. $x + 2y + 8 = 0$.
15. $3x - 4y - 12 = 0$.
16. $5x - 2y - 20 = 0$.
17. $4x + 3y + 10 = 0$.
18. $2x - 5y - 12 = 0$.

19. Write the equation of some line which is parallel to the line $y = 3x - 2$. Draw the graph of both equations.

20. Determine m so that the graph of the equation $y = mx$ will pass through $(2, 6)$.

21. Determine m and b so that the graph of $y = mx + b$ will pass through $(1, 3)$ and $(4, -2)$.

72. **The Graphical Representation of the Solution of a System of Linear Equations.** Let us take any two lines

$$a_1x + b_1y = c_1,$$
$$a_2x + b_2y = c_2.$$

Since their point of intersection must lie on each of the lines, its coordinates may be found by solving the system of equations. Thus the graphical representation of the solution of a system of linear equations in two variables is the point of intersection of the two lines represented by the equations.

We have seen (§ 11) that no solution exists if

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

In this case $a_1b_2 - a_2b_1 = 0$; that is, $a_1/b_1 = a_2/b_2$. Now the slopes of the two lines are given by

$$m_1 = -\frac{a_1}{b_1}, \qquad m_2 = -\frac{a_2}{b_2}.$$

The lines are, therefore, parallel.

If

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{c_1}{c_2},$$

the lines coincide and the coordinates of every point on either line will be a solution of the system.

We may now summarize the foregoing results as follows: *The graph of a single linear equation is a straight line and the graph of a pair of linear equations is the point of intersection of the two lines represented by the equations.*

The student should compare this statement with that of § 62, where his attention was called to the fact that the graph of a single real number is a vector (a line) and the graph of a pair of real numbers is a point.

EXERCISES

Plot the following pairs of lines and find their points of intersection:

1. $\begin{cases} 2x - y = 4, \\ 3x + 2y = 13. \end{cases}$
2. $\begin{cases} x + 2y + 6 = 0, \\ 3x - y + 11 = 0. \end{cases}$
3. $\begin{cases} x + 2y - 6 = 0, \\ 3x - 4y + 17 = 0. \end{cases}$
4. $\begin{cases} 4x + y - 2 = 0, \\ 6x + 7y + 19 = 0. \end{cases}$
5. $\begin{cases} 3x - 5y = 30, \\ 15y - 9x + 45 = 0. \end{cases}$
6. $\begin{cases} 2y = 8x - 9, \\ 16x - 4y - 18 = 0. \end{cases}$
7. $\begin{cases} x + 5y + 11 = 0, \\ 2x - 3y - 17 = 0. \end{cases}$
8. $\begin{cases} 3x + 5y - 61 = 0, \\ 10x - y - 62 = 0. \end{cases}$
9. $\begin{cases} 5x + 4y - 10 = 0, \\ 6x - 3y + 27 = 0. \end{cases}$
10. $\begin{cases} 3x - 6y - 40 = 0, \\ x - 2y - 5 = 0. \end{cases}$
11. $\begin{cases} 5x + y - 11 = 0, \\ 10x - 7y - 58 = 0. \end{cases}$
12. $\begin{cases} 2x + 3y - 27 = 0, \\ 5x - 2y - 1 = 0. \end{cases}$
13. $\begin{cases} 7x + 3y - 23 = 0, \\ 5x + 3y - 19 = 0. \end{cases}$
14. $\begin{cases} x + 4y - 31 = 0, \\ 4x + y - 19 = 0. \end{cases}$
15. $\begin{cases} x + 3y - 7 = 0, \\ x - 2y - 2 = 0. \end{cases}$
16. $\begin{cases} 5x - 3y = 27, \\ 7y - 3x = 15. \end{cases}$
17. $\begin{cases} 5x - 3y - 19 = 0, \\ 7x + 4y - 2 = 0. \end{cases}$
18. $\begin{cases} 2x - 3y - 19 = 0, \\ 7x + 4y - 23 = 0. \end{cases}$
19. $\begin{cases} 7x - 9y - 15 = 0, \\ 8y - 5x + 17 = 0. \end{cases}$
20. $\begin{cases} x + 2y - 11 = 0, \\ 3x + 5y - 29 = 0. \end{cases}$
21. $\begin{cases} 2x - 5y + 16 = 0, \\ 3x + 7y - 5 = 0. \end{cases}$
22. $\begin{cases} 6x - y - 27 = 0, \\ 3x - 8y - 36 = 0. \end{cases}$
23. $\begin{cases} 7x + 2y - 47 = 0, \\ 5x - 4y - 1 = 0. \end{cases}$
24. $\begin{cases} 2x - 5y - 1 = 0, \\ 7x + 3y - 24 = 0. \end{cases}$

CHAPTER IX

QUADRATIC FUNCTIONS

73. The Graph of a Quadratic Function. The most general expression of the second degree in the variable x may be written in the form

$$f(x) = ax^2 + bx + c,$$

where a, b, and c are constants and $a \neq 0$. Such an expression is called a ***quadratic function*** of x.

Consider the quadratic function

$$f(x) = x^2 - 2x + 3.$$

For every assigned value of x there is a corresponding value of $f(x)$, as shown in the following table.

x	0	1	2	3	4	5	-1	-2	-3
$f(x)$	3	2	3	6	11	18	6	11	18

Let us now plot these points and draw a smooth curve through them (Fig. 16). Several important properties of this curve may be derived from the equation

$$y = x^2 - 2x + 3 = (x - 1)^2 + 2.$$

(1) Since $y = 2$ when $x = 1$, and $y > 2$ for all other values of x, it is evident that the point (1, 2) is the lowest point on the curve.

(2) Suppose $y = k$, where $k > 2$. Then $x = 1 \pm \sqrt{k - 2}$. Thus every horizontal line which is more than 2 units above

the x axis cuts the curve in two points which are equidistant from, and on opposite sides of, the vertical line $x - 1 = 0$ which passes through its lowest point.

(3) As x increases (or decreases) indefinitely, y remains positive and increases indefinitely. Hence the curve has two infinite arms which recede indefinitely from the x axis as x increases or decreases.

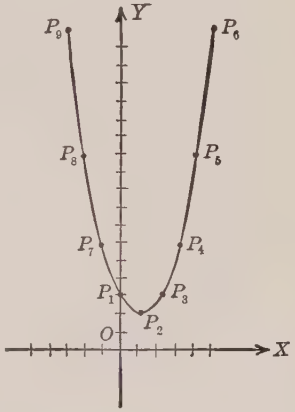


FIG. 16

The graph of every quadratic function is a long smooth curve of the type shown in Fig. 16, the position and shape of the curve depending upon the values of the constants a, b, and c. If a is positive, the curve opens upward; if a is negative, it opens downward. Such a curve is called a ***parabola.*** The lowest (or highest) point is called the ***vertex,*** and the vertical line through the vertex is known as the ***axis*** of the parabola.

74. **Graphical Solution of a Quadratic Equation.** An approximation to the roots of the quadratic equation

$$ax^2 + bx + c = 0$$

may be found graphically as follows.

(1) If the parabola crosses the x axis, the equation has two real and unequal roots, which are the abscissas of the two points where the curve crosses the axis.

(2) If the parabola touches the x axis, the equation has two real and equal roots, each of which is the abscissa of the point where the curve touches the axis.

(3) If the parabola does not cross or touch the x axis, the equation has two conjugate complex roots of the form $p \pm iq$, where p is the abscissa of the vertex of the curve.

EXERCISES

Draw the graphs of the following quadratic functions. Find the zeros of the functions from the graph when possible.

1. $y = x^2 + 3x - 4$.
2. $y = x^2 - 5x + 6$.
3. $y = x^2 + 4x$.
4. $y = x^2 - 4x + 6$.
5. $y = 4 + 3x - x^2$.
6. $y = 6 - x - 2x^2$.
7. $y = 2x - 6x^2$.
8. $y = 3x^2 - 11x - 4$.
9. $y = x^2 - 6x + 9$.
10. $y = 4x - 4x^2 - 1$.
11. $y = x^2 - 4$.
12. $y = 2x^2 - 7x + 3$.
13. $y = 12 + x - x^2$.
14. $y = 7 - 6x - x^2$.
15. $y = x^2 + 2x - 15$.
16. $y = 6 + x - x^2$.

75. **Slope of a Curve.** We have defined (§ 71) the slope (steepness) of a line as the ratio of the change in the ordinate to the change in the abscissa as a point moves along the line. This ratio, which we have called m, is constant.

For example, in the case of the straight line (§ 69) which is the graph of the equation $y = 2x$, when $x = 1$, $y = 2$; when $x = 2$, $y = 4$; and when $x = 4$, $y = 8$. Thus if we increase x by 1, y will be increased by 2, and if we increase x by 3, y will be increased by 6, but the ratio of the increase in y to the corresponding increase in x remains constant and equal to 2.

This increase in a variable as it passes from one value to another is called the ***increment*** of the variable. The increment is generally denoted by writing the Greek letter Δ before the variable.* We read Δx as "the increment of x" or simply "delta x." If the variable decreases, the increment is negative.

In the above example $\Delta x = 1$, $\Delta y = 2$ and $\Delta x = 3$, $\Delta y = 6$, and in either case $m = \Delta y/\Delta x = 2$.

* The student should think of Δx as a single algebraic symbol and not as the product of two symbols Δ and x.

The ***slope of a curve*** at any point is defined as the slope of the tangent to the curve at that point. The slope is not constant as in the case of a straight line but changes from point to point. Let $P(x, y)$ be any point on the curve whose equation is $y = f(x)$. If x increases from OM to ON, y will change from MP to NQ. That is, $PR = \Delta x$, $RQ = \Delta y$. Draw PQ. Then

$$\text{slope of secant } PQ = \frac{RQ}{PR} = \frac{\Delta y}{\Delta x}.$$

The tangent PT is the limiting position of the secant PQ as the point Q approaches the point P along the curve; that is, as Δx approaches zero. Hence the slope of the tangent PT is the limiting value of the slope of the secant. Thus

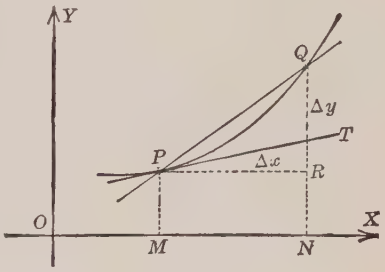


FIG. 17.

$$\text{slope of curve at } P(x, y) = \text{limiting value of } \frac{\Delta y}{\Delta x},$$

as Δx approaches zero.

The problem of finding the slope of a curve at any point is, therefore, reduced to that of finding the limit of $\Delta y/\Delta x$ at that point. Suppose the curve is defined by the equation

$$y = x^2,$$

and that the slope at the point (1, 1) is desired. It is necessary to compute the limiting value of $\Delta y/\Delta x$ at the point where $x = 1$. Let us first compute Δy and $\Delta y/\Delta x$ for several values of Δx.

If $\Delta x =$	then $\Delta y =$	and $\dfrac{\Delta y}{\Delta x} =$
3.	15.	5.
2.	8.	4.
1.	3.	3.
0.1	0.21	2.1
0.01	0.0201	2.01
0.001	0.002001	2.001
Δx	$2\Delta x + (\Delta x)^2$	$2 + \Delta x$

From a study of these results we see that, in this example, as Δx approaches zero, Δy also approaches zero, but the ratio $\Delta y/\Delta x$, instead of approaching zero, approaches 2.

Without restricting ourselves to any one initial value of x, let us compute the general values of Δy and $\Delta y/\Delta x$ when $y = x^2$. Increase x by Δx. Then y will be increased by Δy. Thus

$$y + \Delta y = (x + \Delta x)^2.$$

Hence

$$\Delta y = (x + \Delta x)^2 - x^2 = 2x\Delta x + (\Delta x)^2,$$

and

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x.$$

As Δx approaches zero, the ratio $\Delta y/\Delta x$ approaches $2x$ as a limit. This limiting value of $\Delta y/\Delta x$ is called the ***derivative*** of y with respect to x and is denoted by dy/dx. Thus

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

When $y = x^2$, $dy/dx = 2x$. When $x = 1$, the value of the derivative of y with respect to x is 2. Thus the slope of the parabola $y = x^2$ at the point (1, 1) is 2.

The process of finding the derivative of a function is called ***differentiation.***

The student has already met the idea of limits in the study of elementary mathematics. For example,

(1) the limiting value of the recurring decimal $0.3333\cdots$ is $1/3$;

(2) the limiting value of the recurring decimal $0.9999\cdots$ is unity;

(3) the perimeter of an inscribed regular polygon approaches as a limit the circumference of the circle, as the number of sides is indefinitely increased;

(4) the limiting value of the sum of the first n terms of a G. P. as n is indefinitely increased is $a/(1-r)$, if $-1 < r < 1$.

EXAMPLE. Find the slope of the parabola

$$y = 2x^2 - 4x + 3$$

at the point (2, 3).

Increase x by Δx. Then

$$y + \Delta y = 2(x + \Delta x)^2 - 4(x + \Delta x) + 3.$$

Hence
$$= 2x^2 - 4x + 3 + 4x\Delta x + 2(\Delta x)^2 - 4\Delta x.$$

$$\Delta y = (4x - 4)\Delta x + 2(\Delta x)^2,$$

and

$$\frac{\Delta y}{\Delta x} = 4x - 4 + 2\Delta x.$$

Let Δx approach zero. Then

$$\frac{dy}{dx} = 4x - 4.$$

When $x = 2$, $dy/dx = 4$. Thus the slope of the parabola at the point (2, 3) is 4.

EXERCISES

Find the slope of each of the following parabolas at the point indicated.

1. $y = 3x^2 + x - 1$, (1, 3).

2. $y = x^2 - 8x + 4$, (2, -8).

3. $y = -x^2 + 2x + 6$, (2, 6).

4. $y = -x^2 + 5x$, (0, 0).

5. $y = 4x^2 - 3x - 24$, (3, 3).

6. $y = -2x^2 + 3x + 1$, (1, 2).

7. By making use of the expression for the derivative, show that the parabola $y = x^2 - 2x + 3$ (Fig. 16) is three times as steep at the point (4, 11) as it is at the point (2, 3). Compare the steepness at the points (4, 11) and (-2, 11).

76. The Derivative of a Polynomial. The method of differentiation used in § 75 frequently becomes laborious. In order to reduce the work to a mere matter of routine we shall derive a formula for the differentiation of a general polynomial of which the linear and quadratic functions previously considered are special cases.

The most general polynomial in x may be written in the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

where n is a positive integer and a_0, a_1, a_2, $\cdots$ a_{n-1}, a_n are given constants. Let y be equal to this polynomial. Increase x by Δx; then y is increased by Δy. Thus

$$y + \Delta y = a_0(x + \Delta x)^n + a_1(x + \Delta x)^{n-1} + a_2(x + \Delta x)^{n-2} + \cdots + a_{n-1}(x + \Delta x) + a_n.$$

Expand each of the binomials in the right member and rearrange the terms according to ascending powers of Δx. We find

$$\begin{aligned} y + \Delta y = {} & a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n \\ & + [na_0x^{n-1} + (n-1)a_1x^{n-2} + (n-2)a_2x^{n-3} \\ & \qquad + \cdots + a_{n-1}]\,\Delta x \\ & + [n(n-1)a_0x^{n-2} + (n-1)(n-2)a_1x^{n-3} \\ & \qquad + (n-2)(n-3)a_2x^{n-4} + \cdots]\frac{(\Delta x)^2}{2} \\ & + \text{terms that have higher powers of } \Delta x \text{ as a factor.} \end{aligned}$$

Since the first line in the right member of this equation is equal to y, these terms cancel the y in the left member. Dividing by Δx gives

$$\begin{aligned} \frac{\Delta y}{\Delta x} = {} & na_0x^{n-1} + (n-1)a_1x^{n-2} + (n-2)a_2x^{n-3} + \cdots + a_{n-1} \\ & + \text{terms containing powers of } \Delta x \text{ as a factor.} \end{aligned}$$

Since every term which contains Δx as a factor will approach zero as a limit when Δx approaches zero, we have derived the following theorem.

The derivative of the polynomial

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$$

is

$$na_0x^{n-1} + (n-1)a_1x^{n-2} + (n-2)a_2x^{n-3} + \cdots + a_{n-1}.$$

This theorem may be stated as follows. *In order to obtain the derivative of a polynomial, multiply each term by the exponent of x in that term and then diminish the exponent by unity.*

Thus the derivative of $5x^4 - 2x^3 + 4x^2 - 3x + 7$ is $20x^3 - 6x^2 + 8x - 3$.

EXERCISES

Find the derivative of each of the following functions.

1. $2x + 3$.
2. $mx + b$.
3. $x^2 - 3x + 2$.
4. $-x^2 + 2x + 4$.
5. $x^3 - 4x^2 + 6x - 10$.
6. $x^3 + 4x^2 - 5x + 2$.
7. $x^6 + 3x^4 - 2x$.
8. $3x^2 - 4x + 7$.
9. $x^{2m} + 4x^m + 2$.
10. $5x^8 - 3x^5 + 2x^2 - 6$.

11. What is the slope of the parabola $y = x^2 - 2x + 5$ at the point (1, 4)? What is the slope at (0, 5)? at (2, 5)?

12. What is the slope of the curve $y = x^3 - 3x$ at (2, 2)?

13. Show that the derivative of any constant is equal to zero.

14. Show by differentiation that the slope of the line $y = mx + b$ is m.

15. At what point on the curve $y = 3x^2 - 4x + 5$ is the ordinate increasing twice as fast as the abscissa?

16. What is the slope of the curve $y = x^3 - 4x^2 + 7x - 5$ at the point (2, 1)?

17. Show that the slope of the curve $y = x^5 + 2x + 1$ is always positive.

18. At what point on the curve $y = x^2 - x + 2$ is the ordinate increasing three times as fast as the abscissa?

77. Maximum and Minimum Values of Quadratic Functions. We have seen that the graph of the general quadratic function

$$y = ax^2 + bx + c$$

is a parabola, which opens upward if a is positive and downward if a is negative. The slope of this parabola, which is given by

$$\frac{dy}{dx} = 2ax + b,$$

is a function of x, and consequently varies from point to point.

If we think of the parabola as traced by a point moving along the curve from left to right, it is obvious that (§ 71) at any given point

(1) the curve is *rising*, if the slope is *positive;*

(2) the curve is *falling*, if the slope is *negative;*

(3) the curve is *parallel to the x axis*, if the slope is *zero.*

The slope of the parabola, which is defined by a linear function of x, will be zero when and only when $x = -b/(2a)$. Consequently, there is one and only one point on the parabola where the tangent is parallel to the x axis. This point is the vertex (§ 73), or turning point, of the parabola. It will either be the lowest, or the highest, point on the curve.

The abscissa of the vertex of the parabola $y = ax^2 + bx + c$ is evidently $x = -b/(2a)$, and its ordinate may be obtained by substituting this value for x in the equation of the parabola. The ordinate thus obtained will either be the maximum, or the minimum, value of the given quadratic function.

A glance at Fig. 18 shows that when the moving point passes through the lowest point on the curve, the slope changes from negative to positive; and when it passes through the highest point, the slope changes from positive to negative.

The rule for determining the maximum (or minimum) value of the quadratic function may now be stated as follows:

Determine the value of x for which the derivative dy/dx is equal to zero. Think of x as increasing through this *critical value.* Then the value of the quadratic function which corresponds to this value of x will be

(1) a *maximum,* if dy/dx changes from $+$ to $-$;

(2) a *minimum,* if dy/dx changes from $-$ to $+$.

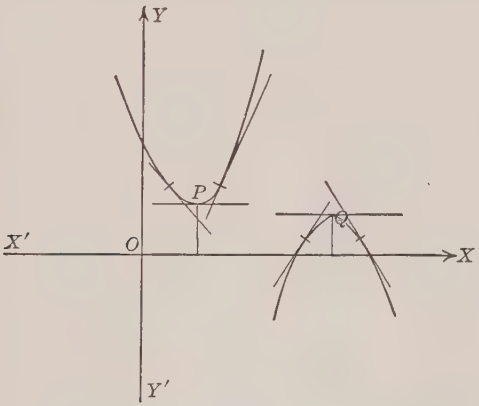


FIG. 18

EXAMPLE 1. Find the maximum (or minimum) value of the function (Fig. 16) $y = x^2 - 2x + 3$.

We find

$$\frac{dy}{dx} = 2x - 2,$$

which is equal to zero when $x = 1$. When x is less than 1 the slope is negative and when x is greater than 1 the slope is positive. That is, as x increases through the critical value ($x = 1$), the derivative changes from $-$ to $+$. Thus the function takes on its minimum value when $x = 1$. Substituting $x = 1$ in $x^2 - 2x + 3$ gives $y = 2$, the minimum value of the function. The function has no maximum value.

EXAMPLE 2. Divide 12 into two parts such that their product shall be a maximum.

Let x = one part. Then $12 - x$ = the other part, and $x(12 - x)$ = their product.

Let $y = x(12 - x) = 12x - x^2$. Then $dy/dx = 12 - 2x$, and the critical value of x is $x = 6$. As x increases through this value, the derivative changes from $+$ to $-$. Hence $x = 6$ makes the product a maximum. Thus we must divide 12 into two equal parts in order to obtain the maximum product.

EXERCISES

Find the maximum or minimum value of each of the following quadratic functions.

1. $x^2 - 4x - 3$.
2. $2x^2 + 8x - 10$.
3. $2 + 6x - x^2$.
4. $x^2 + 3$.
5. $x^2 - 4x + 4$.
6. $-x^2 + 6x - 9$.
7. $4x^2 - 16x$.
8. $8x - x^2$.

9. Show that the equation of the axis (§ 73) of the parabola $y = 2x^2 - 8x + 3$ is $x = 2$.

10. Show that the equation of the axis of the parabola $y = ax^2 + bx + c$ is $x + b/(2a) = 0$.

11. By substituting $-b/(2a) + k$ and $-b/(2a) - k$ for x in the expression for the slope of the parabola, $y = ax^2 + bx + c$, show that for points equidistant from the axis the slope of the parabola has the same value but opposite sign.

12. Of all rectangles having a perimeter of 40 inches, which one has the greatest area?

13. A man wishes to fence off a rectangular piece of ground. What should the dimensions be in order that as much ground as possible may be enclosed with 200 rods of fence?

14. Is it possible to have a rectangular field that contains 75000 square yards and has a perimeter of 1000 yards?

15. What is the maximum height (Ex. 11, p. 71) reached by a ball thrown up with an initial velocity of 100 feet per second?

CHAPTER X

SIMULTANEOUS QUADRATIC EQUATIONS

78. **Quadratic Equations in Two Variables.** In Chapter VIII it was shown that every equation of the first degree in two variables represents a straight line, and that the solution of a system of two linear equations may be obtained graphically by drawing the two lines represented by the equations and then observing the x and the y of their point of intersection. This same method may be used when one (or both) of the equations is of the second degree, although the graphs are, in general, not lines but curves whose properties we shall now investigate.

Before discussing in detail the curves which are the graphs of different types of equations of the second degree we shall state and prove a general property which is common to all of them.

79. **Symmetry.** A curve is *symmetric with respect to a line* l if every chord of the curve perpendicular to l is bisected by l. If the entire plane is folded along the line l, the part of the curve on one side of l will fall exactly upon the part of the curve on the other side of l.

Suppose an equation contains only the second power of y; that is, suppose the term of the first degree in y is missing. Then for any given value of x, the corresponding values of y will be equal but of opposite sign. When these two points are plotted, it will be seen that the line joining them is perpendicular to, and bisected by, the x axis. The curve which is the graph of such an equation is, therefore, symmetric with respect to the x axis.

Thus in the equation $x^2 + y^2 = 25$, when $x = 3$, $y = \pm 4$ and the points (3, 4) and (3, $-$ 4) are on the curve which is the graph of the equation. Since for every value of x between -5 and 5 we obtain a pair of points similarly situated with reference to the x axis, that part of the curve which lies between $x = -5$ and $x = 5$ is symmetric with respect to the x axis.

In a similar manner it may be shown that if an equation contains only the second power of x, the curve which is the graph of the equation will be symmetric with respect to the y axis.

Thus the curve whose equation is $x^2 + y^2 = 25$ is also symmetric with respect to the y axis.

A curve is *symmetric with respect to a point P* if every chord of the curve through that point is bisected by that point. It is obvious that, if a curve is symmetric with respect to each of two lines, it is symmetric with respect to their point of intersection.

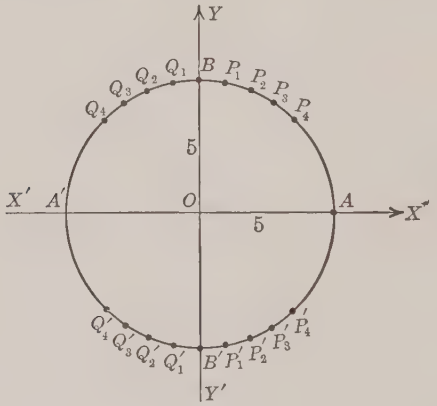


Fig. 19

Thus the curve whose equation is $x^2 + y^2 = 25$ is symmetric with respect to the origin.

80. **The Circle.** Consider the equation

$$x^2 + y^2 = 25.$$

Solving for y, we find

$$y = \pm \sqrt{25 - x^2}.$$

Compute y by assuming values for x, making use of the tables (p. 300). We find the following pairs of values.

	B	P_1	P_2	P_3	P_4	A	Q_1	Q_2	Q_3	Q_4	A'
When $x =$	0	1	2	3	4	5	-1	-2	-3	-4	-5
then $y =$	± 5	± 4.9	± 4.6	± 4	± 3	0	± 4.9	± 4.6	± 4	± 3	0

Since only the squares of x and y occur in the equation, the curve is symmetric with respect to both axes and the origin.

For values of x greater than 5, or less than -5, the expression $25 - x^2$ becomes negative and hence $\sqrt{25 - x^2}$ becomes imaginary. Since only real numbers can represent coordinates of points, no part of the curve can lie to the right of A or to the left of A'.

This curve is a *circle* of radius 5 with its center at the origin (Fig. 19). In fact, every equation of the type $x^2 + y^2 = r^2$ represents a circle with its center at the origin and radius r.

81. **The Ellipse.** Consider the equation

$$9x^2 + 16y^2 = 576.$$

Solving for y, we find

$$y = \pm \tfrac{3}{4}\sqrt{64 - x^2}.$$

Calculate values for y by assuming values for x.

	B	P_1	P_2	P_3	P_4	P_5	P_6	P_7	A
When $x=$	0	1	2	3	4	5	6	7	8
then $y=$	± 6	± 5.9	± 5.8	± 5.6	± 5.2	± 4.7	± 3.9	± 2.9	0

		Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	A'
When $x=$		-1	-2	-3	-4	-5	-6	-7	-8
then $y=$		± 5.9	± 5.8	± 5.6	± 5.2	± 4.7	± 3.9	± 2.9	0

For values of x greater than 8 or less than -8 the values of y are imaginary, and for values of y greater than 6 or less than -6 the values of x are imaginary. The curve lies, therefore, entirely within the rectangle indicated in the figure. Since only even powers of x and y occur in the given equation, the curve is symmetric with respect to the two axes and, therefore, with respect to the origin.

It will be found that all the points whose coordinates are given above lie on the oval-shaped curve shown in Fig. 20. This curve is an ***ellipse.***

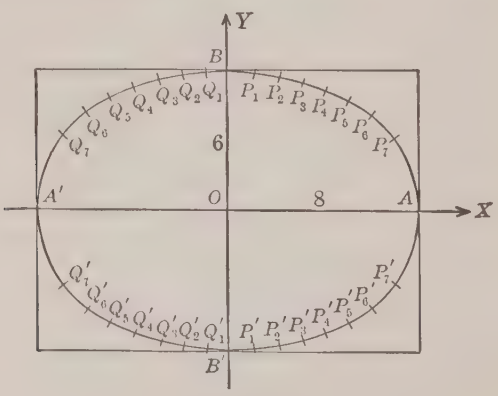


FIG. 20

The student should note the difference between the equation of an ellipse and the equation of a circle. If the coefficients of x^2 and y^2 in the above equation had been equal, the curve would have been a circle.

82. **The Hyperbola.** Consider the equation

$$9x^2 - 16y^2 = 144.$$

Solving for y, we find

$$y = \pm \tfrac{3}{4}\sqrt{x^2 - 16}.$$

Calculate values for y by assuming values for x.

	A	P_1	P_2	P_3	A'	Q_1	Q_2	Q_3
When $x =$	4	5	6	7	-4	-5	-6	-7
then $y =$	0	± 2.2	± 3.4	± 4.3	0	± 2.2	± 3.4	± 4.3

Since $\sqrt{x^2 - 16}$ is imaginary for every value of x between -4 and $+4$, there are no points on the curve between the vertical lines through the points A and A'. Since only even powers of x and y occur in the equation, the curve is symmetric with respect to both axes, and, therefore, with respect to the origin.

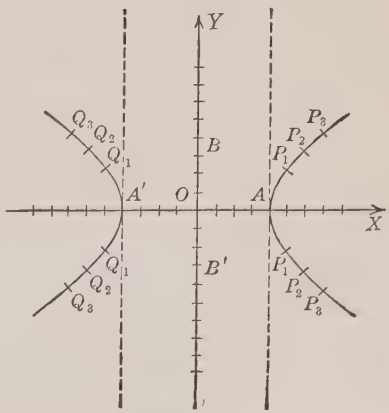


FIG. 21

It will be found that all the points whose coordinates are given above lie on the curve shown in Fig. 21. This curve, which is called a ***hyperbola,*** consists of two separate branches.

Let us now consider the equation

$$xy = 1.$$

Solving for y, we find

$$y = 1/x.$$

Compute values for y by assuming values for x.

	P_1	P_2	P_3	P_4	P_5	P_6	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
When x =	1	2	3	4	0.5	0.25	−1	−2	−3	−4	−0.5	−0.25
then y =	1	0.50	0.33	0.25	2	4	−1	−0.50	−0.33	−0.25	−2	−4

If the point (x_1, y_1) is on the curve, the point $(-x_1, -y_1)$ is also on the curve. That is, the curve is symmetric with respect to the origin. It is, therefore, only necessary to discuss the graph for positive values of x. For small values of x the values of y become large, and for small values of y the values of x become large. That is, the curve approaches the two axes as shown in Fig. 22.

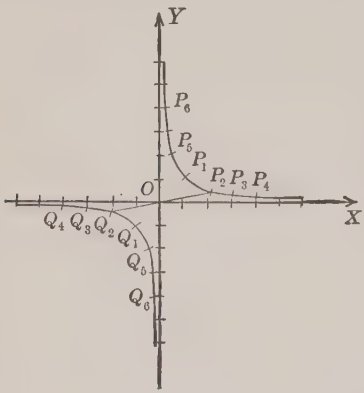


FIG. 22

This curve is a ***hyperbola*** similar to the one in Fig. 21, but differently situated with reference to the two axes.

This is an example of a curve which is symmetric with respect to the origin but not with respect to either axis.

83. **The Parabola.** The student has already learned (§ 73) that the graph of the equation

$$y = ax^2 + bx + c$$

is a ***parabola.*** Suppose we interchange the variables x and y and write the equation in the form

$$x = ay^2 + by + c.$$

Then this equation will represent a parabola whose axis (§ 73) is parallel to the x axis and which opens to the right when a is positive and to the left when a is negative.

84. **Conics.** The ellipse (including the circle as a special case), the hyperbola, and the parabola are very useful in the study of mathematics and are grouped under the general name of ***conic sections,*** or simply ***conics.*** It may be shown that the graph of every equation of the second degree in two variables is, in general, one of these curves. The position of the conic with reference to the x and y axes depends upon the coefficients in the given equation.

One apparent exception should be mentioned. If the left member of the equation, after all terms have been transposed to that member may be written as the product of two linear factors, the graph consists of *two straight lines* and not a conic.

Thus the graph of the equation $4x^2 - 9y^2 = 0$ consists of the two lines $2x + 3y = 0$ and $2x - 3y = 0$.

85. **Graphical Solution.** The method of solving a system of two non-linear equations in two variables by means of graphs is similar to § 72 and is illustrated by the examples below. It is obvious that such a system may consist of one linear and one quadratic equation, or of two quadratic equations. We shall consider the two cases separately.

(1) *One Equation Linear and the Other Quadratic.* In this case the graph of one equation will be a straight line and the graph of the other equation will be a conic (§ 84). The line may cross the conic in two points, touch the conic, or lie entirely outside the conic. Thus the system of equations

may have two different solutions, two equal solutions, or no solution.

EXAMPLE. Solve the system

$$2x - y - 1 = 0,$$
$$4x^2 + y^2 = 25.$$

The first equation represents a straight line which may be plotted immediately by finding the coordinates of the two points where it crosses the axes.

The second equation represents an ellipse (§ 81). If we assign different values to x and then calculate the corresponding values of y, we obtain the following table.

When $x=$	0	1	2	2.5
then $y=$	± 5	± 4.6	± 3	0

These points will lie on the ellipse shown in Fig. 23.

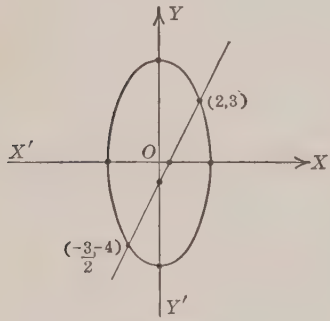


FIG. 23

The line crosses the ellipse at two points whose coordinates seem to be (2, 3) and (− 3/2, − 4). Hence the solution of the system of equations consists of two pairs of numbers $x = 2$, $y = 3$ and $x = -3/2$, $y = -4$.

CHECK. Each pair of numbers should now be substituted in both of the given equations to see whether both equations are satisfied.

For the solution $x = 2$, $y = 3$, we have $2x - y - 1 = 4 - 3 - 1 = 0$, and $4x^2 + y^2 = 16 + 9 = 25$, as required.

For the solution $x = -3/2$, $y = -4$, we have $2x - y - 1 = -3 + 4 - 1 = 0$, and $4x^2 + y^2 = 9 + 16 = 25$, as required.

(2) *Both Equations Quadratic.* In this case the graph of each equation will be a conic. Since two conics may intersect in as many as four points, we may expect to find in some cases as many as four solutions of such a system of equations.

EXAMPLE. Solve the system

$$x^2 + y^2 = 34,$$
$$xy = 15.$$

The first equation represents a circle (§ 80) whose center is at the origin and whose radius is $\sqrt{34} = 5.8$

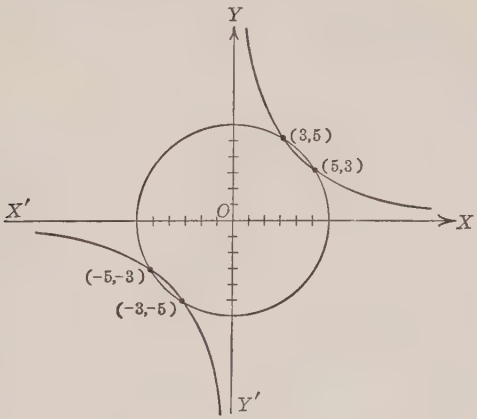


FIG. 24

The second equation represents a hyperbola (§ 82). If we assign different values to x and then calculate the corresponding values of y, we find:

When $x =$	1	3	5	7	9	11	13	15
then $y =$	15	5	3	2.1	1.7	1.4	1.2	1

The circle and the hyperbola may now be drawn as in Fig. 24. The two curves intersect in four points whose coordinates seem to be (5, 3), (3, 5), (− 5, − 3), and (− 3, − 5). Hence the solution of the system of equations consists of the four pairs of numbers $x = 5$, $y = 3$; $x = 3$, $y = 5$; $x = -5$, $y = -3$; and $x = -3$, $y = -5$.

CHECK. It is obvious that each of these four pairs of numbers will satisfy both equations. Hence the solution is correct.

EXERCISES

Draw the graphs of the following systems and then determine the solutions whenever they are real.

1. $\begin{cases} x^2 = 2y, \\ y = 2x. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 106, \\ x + y = 14. \end{cases}$

3. $\begin{cases} x - y = 4, \\ xy = 60. \end{cases}$

4. $\begin{cases} 2x + y = 14, \\ xy = -16. \end{cases}$

5. $\begin{cases} y = 4x^2 + 2x + 1, \\ 2x + y - 1 = 0. \end{cases}$

6. $\begin{cases} 3x - 5y^2 + 116 = 0, \\ 7x + 4y^2 - 121 = 0. \end{cases}$

7. $\begin{cases} 4x^2 + 5y^2 = 109, \\ 3x^2 - 2y^2 = 30. \end{cases}$

8. $\begin{cases} x^2 + 4y^2 = 16, \\ 9x^2 - 16y^2 + 144 = 0. \end{cases}$

86. **Solution by Elimination.** In § 10 we derived a method for the solution of a system of linear equations by eliminating one of the unknown quantities and then solving a linear equation in the other unknown. The application of this method to certain systems of quadratic equations will be illustrated by the examples which follow. As in § 85 we shall consider two types of systems.

(1) *One Equation Linear and the Other Quadratic.* In this case the system may always be solved by eliminating one unknown and then solving a quadratic equation.

EXAMPLE 1. Solve the system

(1) $2x - y - 1 = 0,$

(2) $4x^2 + y^2 = 25.$

Solving (1) for y, we find

$$y = 2x - 1. \tag{3}$$

Eliminating y from (2) by substituting $2x - 1$ for y in (2), we obtain

$$4x^2 + (2x - 1)^2 = 25,$$

or, after reduction,

$$2x^2 - x - 6 = 0. \tag{4}$$

Solving (4) for x gives

$$x = 2 \quad \text{and} \quad x = -3/2.$$

When $x = 2$ it follows from (3) that $y = 3$. Also when $x = -3/2$ we find from (3) that $y = -4$. Hence the two solutions are $x = 2$, $y = 3$; and $x = -3/2$, $y = -4$. Observe that these are the same results we obtained graphically in § 85.

Example 2. Solve the system

$$3x - 2y - 5 = 0, \tag{1}$$

$$x^2 - y^2 = 5. \tag{2}$$

Solving (1) for y, we find

$$y = \frac{3x - 5}{2}. \tag{3}$$

Substituting this value of y in (2), we have

$$x^2 - \left(\frac{3x - 5}{2}\right)^2 = 5,$$

or, after reduction,

$$x^2 - 6x + 9 = 0. \tag{4}$$

This equation has two equal roots

$$x = 3, \quad \text{and} \quad x = 3.$$

Substituting 3 for x in (3), we find $y = 2$. Thus the system has only one solution $x = 3$, $y = 2$. This means that the line represented by equation (1) is tangent to the hyperbola represented by (2) at the point (3, 2).

Check. Substitute $x = 3$ and $y = 2$ in the given equations. We find, from (1), $9 - 4 - 5 = 0$, and from (2), $9 - 4 = 5$.

EXAMPLE 3. Solve the system

$$x + y = 5, \tag{1}$$

$$4x^2 + 9y^2 = 36. \tag{2}$$

Solving (1) for y, we find

$$y = 5 - x. \tag{3}$$

Substituting this value of y in (2), we have

$$4x^2 + 9(5 - x)^2 = 36,$$

or, after reduction,

$$13x^2 - 90x + 189 = 0.$$

Solving for x by the formula (§ 20) gives

$$x = \frac{45 \pm 12i\sqrt{3}}{13}.$$

Substituting these values for x in (3) gives

$$y = 5 - \frac{45 \pm 12i\sqrt{3}}{13} = \frac{20 \mp 12i\sqrt{3}}{13}.$$

Hence the two solutions are

$$x = \frac{45 + 12i\sqrt{3}}{13}, \qquad y = \frac{20 - 12i\sqrt{3}}{13},$$

and

$$x = \frac{45 - 12i\sqrt{3}}{13}, \qquad y = \frac{20 + 12i\sqrt{3}}{2}.$$

Since both solutions are imaginary, the line represented by equation (1) will not cross or touch the ellipse represented by (2).

CHECK. The student should check these solutions by substituting in (2).

(2) *Both Equations Quadratic.* When both equations are of the second degree, the process of elimination will, in general, produce an equation of the fourth degree. It will, therefore, not be possible to solve every system of simultaneous quadratic equations until we have made a study of equations of degree higher than the second.

In certain cases, however, the solution may be found.

EXAMPLE 4. Solve the system

$$2x^2 + 5y^2 = 37. \tag{1}$$

$$3x^2 - 4y^2 = 44. \tag{2}$$

Multiplying (1) by 3 and (2) by 2, we have

$$6x^2 + 15y^2 = 111, \tag{3}$$

$$6x^2 - 8y^2 = 88. \tag{4}$$

Subtracting (4) from (3), we find

$$23y^2 = 23.$$

Hence $y^2 = 1$ and $y = \pm 1$. Substituting 1 for y in (1) gives $2x^2 + 5 = 37$, or $x = \pm 4$. Substituting -1 for y in (1) gives $2x^2 + 5 = 37$, or $x = \pm 4$. Hence the four solutions are

$$x = 4, y = 1; x = 4, y = -1; x = -4, y = 1; x = -4, y = -1.$$

Thus the ellipse (1) and the hyperbola (2) intersect in four points represented by the above sets of coordinates.

EXAMPLE 5. Solve the system

$$2x^2 + xy - y^2 = 0, \tag{1}$$

$$2x^2 + y = 3. \tag{2}$$

Equation (1) may be written in the form

$$(2x - y)(x + y) = 0.$$

Hence, either

$$2x - y = 0, \quad \text{or} \quad x + y = 0.$$

Thus the given system may be replaced by the two systems:

$$\begin{aligned} 2x - y &= 0, \\ 2x^2 + y &= 3, \end{aligned} \qquad \text{and} \qquad \begin{aligned} x + y &= 0, \\ 2x^2 + y &= 3. \end{aligned}$$

Each of these new systems may now be solved by the method used in Example 1 above. We find that the solutions of the first system are

$$x = \frac{-1 + \sqrt{7}}{2}, \qquad y = -1 + \sqrt{7},$$

and

$$x = \frac{-1-\sqrt{7}}{2}, \qquad y = -1-\sqrt{7},$$

and the solutions of the second system are

$$x = -1, \quad y = 1, \qquad \text{and} \qquad x = 3/2, \quad y = -3/2.$$

These four solutions are, therefore, the solutions of the given system.

NOTE. This method may always be used when one (or both) of the equations is of the form $ax^2 + bxy + cy^2 = 0$.

EXAMPLE 6. Solve the system

(1) $$x^2 + y^2 = 45,$$

(2) $$xy = 18.$$

Multiplying (2) by 2, we have

(3) $$2xy = 36.$$

Adding (3) to (1), then subtracting (3) from (1), we find

(4) $$x^2 + 2xy + y^2 = 81,$$

(5) $$x^2 - 2xy + y^2 = 9.$$

Whence

$$x + y = \pm 9, \qquad x - y = \pm 3.$$

Thus the given system may be replaced by the four systems:

$$\begin{cases} x+y=9, \\ x-y=3, \end{cases} \quad \begin{cases} x+y=9, \\ x-y=-3, \end{cases} \quad \begin{cases} x+y=-9, \\ x-y=3, \end{cases} \quad \begin{cases} x+y=-9, \\ x-y=-3. \end{cases}$$

The solutions of these systems, in order, are

$$x = 6, y = 3; \; x = 3, y = 6; \; x = -6, y = -3; \; x = -3, y = -6.$$

These four solutions are, therefore, the solutions of the given system.

NOTE. This method may always be used when both of the equations are symmetrical with respect to x and y; that is, when x and y can be interchanged without changing the equations.

EXAMPLE 7. Solve the system

(1) $$2x^2 + 7xy + 4y^2 = 2,$$

(2) $$3x^2 + 8xy - 4y^2 = -72.$$

Introducing a third unknown number v defined by the equation

$$y = vx, \tag{3}$$

and substituting vx for y in (1) and (2), we find

$$2x^2 + 7vx^2 + 4v^2x^2 = 2, \tag{4}$$

or

$$x^2 = \frac{2}{2 + 7v + 4v^2};$$

and

$$3x^2 + 8vx^2 - 4v^2x^2 = -72, \tag{5}$$

or

$$x^2 = \frac{-72}{3 + 8v - 4v^2}.$$

Equating the two values of x^2, we have

$$\frac{2}{2 + 7v + 4v^2} = \frac{-72}{3 + 8v - 4v^2},$$

or

$$28v^2 + 52v + 15 = 0. \tag{6}$$

The roots of this equation are $v = -3/2$ and $v = -5/14$. Put $v = -3/2$ in (4). We find $x^2 = 4$; then $x = \pm 2$, and, from (3), we have

$$y = vx = -\frac{3}{2}(\pm 2) = \mp 3.$$

Put $v = -5/14$ in (4). We find $x^2 = 196$; then $x = \pm 14$, and

$$y = vx = -\frac{5}{14}(\pm 14) = \mp 5.$$

Thus the four solutions of the given system are

$$x = 2, \quad y = -3; \qquad x = -2, \quad y = 3;$$
$$x = 14, \quad y = -5; \qquad x = -14, \quad y = 5.$$

NOTE. This method may always be used when neither equation contains a term of the first degree.

EXERCISES

Solve the following systems of equations:

1. $\begin{cases} 3x^2 - xy - y^2 = -3, \\ 2x - 3y = 5. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 200, \\ x + y = 16. \end{cases}$

3. $\begin{cases} 3x^2 - 9x - y = 2, \\ 3x - y = 2. \end{cases}$

4. $\begin{cases} x^2 + y^2 = 74, \\ x - y = 12, \end{cases}$

5. $\begin{cases} x^2 - 4y^2 + x = 0, \\ 3x + 6y = 1. \end{cases}$

6. $\begin{cases} x^2 + 4xy = 13, \\ 2xy + 9y^2 = 87. \end{cases}$

7. $\begin{cases} 7xy - 8x^2 = 10, \\ 8y^2 - 9xy = 18. \end{cases}$

8. $\begin{cases} 16x^2 - 9y^2 = 0, \\ 2xy - 5x + 6y = 33. \end{cases}$

9. $\begin{cases} x^2 + 3xy + 2y^2 = 15, \\ 4x^2 + 5y^2 = 24. \end{cases}$

10. $\begin{cases} x^2 - 2xy = 0, \\ 2x^2 + xy + y^2 = 44. \end{cases}$

11. $\begin{cases} 3x^2 + xy + y^2 = 15, \\ 3x^2 - 31xy + 5y^2 = -45. \end{cases}$

12. $\begin{cases} x^2 - 7xy + 12y^2 = 0, \\ xy + 3y - 2x = 21. \end{cases}$

13. $\begin{cases} 4x^2 - 6xy + 9y^2 = 73, \\ xy = 12. \end{cases}$

14. $\begin{cases} 4x^2 - 2xy - y^2 = -16, \\ 4x^2 + 7xy + 2y^2 = 104. \end{cases}$

15. $\begin{cases} x^2 + 3xy - 9y^2 = 9, \\ x^2 - 13xy + 21y^2 = -9. \end{cases}$

16. $\begin{cases} x^2 + y^2 + 2x + 2y = 23, \\ xy = 6. \end{cases}$

17. The sum of the squares of two numbers is 145 and twice the product of the numbers is 144. Find the numbers.

18. The difference of the squares of two numbers is 27, and the difference of the numbers is one-third their sum. Find the numbers.

19. The hypotenuse of a right-angled triangle is 20 feet, and the area is 96 square feet. Find the length of the other two sides.

20. The difference of two numbers is 32, and their product is 833. What are the numbers?

21. The fence around a rectangular field is 2800 feet long. The diagonal of the field is 1000 feet long. What are the dimensions of the field?

22. Find two numbers whose difference is 19 and whose product is 216.

23. A broker bought a certain number of shares of stock for \$1875; he sold all but 15 of them for \$1740, gaining \$4.00 on each share. How many shares did he buy?

24. The diagonal of a rectangle is 20 feet. If the length be increased by 8 feet and the breadth be diminished by 5 feet, the diagonal will be 25 feet. What are the dimensions of the rectangle?

25. If the length and breadth of a rectangle were each increased by 1, the area would be 99, and if they were each diminished by 1, the area would be 63. Find the dimensions.

26. A rectangle is surrounded by a border whose width is 5 inches. The area of the rectangle is 168 square inches, that of the border 360 square inches. Find the length and breadth of the rectangle.

27. The three sides of a right-angled triangle are 18, 24, and 30, respectively. Find the length of the perpendicular from the right angle to the hypotenuse.

28. The product of two numbers is 60 more than ten times the first number, and 30 more than ten times the second number. Find the numbers.

29. The difference of two numbers which are formed by the same two digits is 18/55 of their sum; the difference of the squares of the numbers is 3960. Find the numbers.

30. The sum of the squares of two numbers plus five times their product is equal to 445; and the sum of their squares minus their product is equal to 67. Find the numbers.

31. The square of the sum of two numbers exceeds twenty times the sum of the numbers by 156. The difference of the numbers is 4. Find the numbers.

32. If a number of two digits be divided by the product of the digits, the quotient will be 3. If 18 be added to the number, the digits will be interchanged. What is the number?

33. B can do a piece of work in two hours less time than A, and together they can do the work in 2 hours and 24 minutes. How long does each alone take to do the work?

CHAPTER XI

THEORY OF EQUATIONS

87. **Polynomial Functions.** The function of x of the form

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

where a_0, a_1, a_2, $\cdots$, a_n are constants and n is some positive integer, is called a *polynomial function* in x of degree n.

If $n = 1$, $f(x) = a_0x + a_1$, a linear function in x.

If $n = 2$, $f(x) = a_0x^2 + a_1x + a_2$, a quadratic function in x.

When $n = 3$, we have a cubic function in x; and so on.

If $f(x) = 0$, that is, if

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0,$$

we have a polynomial equation of degree n if $a_0 \neq 0$. Such an equation is called an *integral rational equation* in x.

Any value of x that makes $f(x) = 0$ is a root of this equation.

88. **The Graph of a Polynomial Function.** As in §§ 71 and 73, there is one and only one value of $f(x)$ corresponding to any given value of x. If a table of values be computed and the pairs of numbers be plotted, a smooth curve can be drawn through the points, thus forming the graph of the polynomial function as in the examples following.

EXAMPLE 1. Draw the graph of the function

$$y = f(x) = x^3 - 3x^2 - 6x + 8.$$

For every assigned value of x there is a corresponding value of $f(x)$ as shown in the following table.

If these points be plotted and a smooth curve be drawn through them, we obtain the curve shown in Fig. 25. The graph of every

cubic function is an indefinitely long smooth curve of this same type, crossing the x axis in not more than three points.

x	y
-3	-28
-2	0
-1	10
0	8
1	0
2	-8
3	-10
4	0
5	28

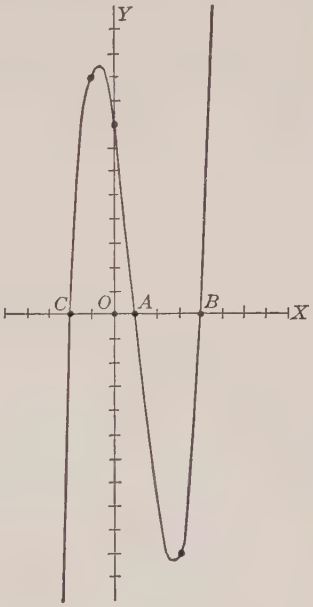


FIG. 25

From the graph we see that the curve crosses the x axis at the points A, B, and C; that is, the values of x that make $y = 0$ are $x_1 = OA$, $x_2 = OB$, and $x_3 = OC$. Thus x_1, x_2, and x_3 are roots of the cubic equation

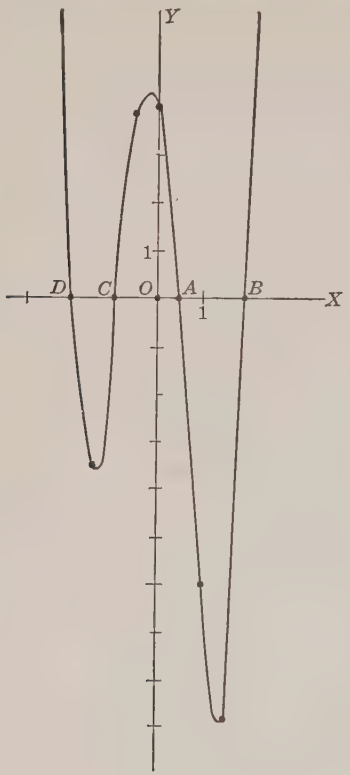
$$x^3 - 3x^2 - 6x + 8 = 0.$$

From the graph we can readily determine the roots of the equation as $x_1 = 1$, $x_2 = 4$, $x_3 = -2$. It is evident that in most cases only a rough approximation to the roots of an equation can be found by plotting the curve. The problem of finding a close approximation to the roots of a rational integral equation in x will be considered later.

EXAMPLE 2. Draw the graph of the function

$$f(x) = 2x^4 + x^3 - 9x^2 - 4x + 4.$$

We find the following pairs of values of x and $f(x)$.



x	y
-2	0
$-\frac{3}{2}$	$-\frac{7}{2}$
-1	0
$-\frac{1}{2}$	$\frac{15}{4}$
0	4
$\frac{1}{2}$	0
1	-6
$\frac{3}{2}$	$-\frac{35}{4}$
2	0

FIG. 26

Fig. 26 shows the graph obtained by plotting these points. The graph of every function of the fourth degree is of this same type, a long smooth curve crossing the x axis in not more than four points.

The roots of the equation

$$2x^4 + x^3 - 9x^2 - 4x + 4 = 0$$

are $x_1 = OA$, $x_2 = OB$, $x_3 = OC$, and $x_4 = OD$.

From the graph it appears that

$$x_1 = 1/2, \qquad x_2 = 2, \qquad x_3 = -1, \qquad x_4 = -2.$$

In a similar manner it may be shown that the graph of any polynomial function of x of degree n is an indefinitely long smooth curve crossing the x axis in not more than n points. If n is even both infinite branches lie on the same side of the x axis, and if n is odd they lie on opposite sides of the x axis.

This statement is obviously true in the case of the straight line and the parabola where $n = 1$ and 2, respectively.

89. **Remainder and Factor Theorems.** In the work which follows we shall find it convenient to make use of two theorems which we shall now prove.

The Remainder Theorem. *If the polynomial function*

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

be divided by $x - r$, the remainder is $f(r)$.

Proof. Let $f(x)$ be divided by $x - r$ and let $Q(x)$ denote the quotient, and let R (which will be free from x) denote the remainder. Then

$$f(x) = (x - r)\cdot Q(x) + R.$$

If we substitute $x = r$, in this identity we have

$$f(r) = R.$$

The Factor Theorem. *If the polynomial function*

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

becomes zero when r is substituted for x [that is, if $f(r) = 0$], then $x - r$ is a factor of $f(x)$.

Proof. Since

$$f(x) = (x - r)\cdot Q(x) + R,$$

and

$$f(r) = R = 0,$$

we have

$$f(x) = (x - r) \cdot Q(x).$$

We may now use synthetic division (§ 4) and the remainder theorem in finding the value of a polynomial function for a given value of x.

Thus if $y = x^3 + 2x^2 - 3x + 2$, we have by synthetic division

$$\begin{array}{r} 1 + 2 - 3 + 2 \,|\, 3 \\ \quad + 3 + 15 + 36 \\ \hline 1 + 5 + 12 + \mathbf{38}, \text{ the remainder.} \end{array}$$

That is, if $x = 3$, $y = 38$.

EXAMPLE. Find the value of the function

$$y = f(x) = x^3 + 3x^2 - 2x - 4$$

when $x = 1, 2, 3, -1, -2$, and -3.

Making use of synthetic division we have

$$\begin{array}{r} 1 + 3 - 2 - 4 \,|\, 1 \\ \quad 1 + 4 + 2 \\ \hline 1 + 4 + 2 - \mathbf{2} \end{array} \qquad \begin{array}{r} 1 + 3 - 2 - 4 \,|\, 2 \\ \quad + 2 + 10 + 16 \\ \hline 1 + 5 + 8 + \mathbf{12} \end{array}$$

$$\begin{array}{r} 1 + 3 - 2 - 4 \,|\, 3 \\ \quad + 3 + 18 + 48 \\ \hline 1 + 6 + 16 + \mathbf{44} \end{array} \qquad \begin{array}{r} 1 + 3 - 2 - 4 \,|\, -1 \\ \quad - 1 - 2 + 4 \\ \hline 1 + 2 - 4 + \mathbf{0} \end{array}$$

$$\begin{array}{r} 1 + 3 - 2 - 4 \,|\, -2 \\ \quad - 2 - 2 + 8 \\ \hline 1 + 1 - 4 + \mathbf{4} \end{array} \qquad \begin{array}{r} 1 + 3 - 2 - 4 \,|\, -3 \\ \quad - 3 + 0 + 6 \\ \hline 1 + 0 - 2 + \mathbf{2} \end{array}$$

Hence we have

$$\begin{array}{ll} x = & -3, -2, -1, \quad 1, \quad 2, \quad 3. \\ y = & \quad 2, \quad 4, \quad 0, -2, 12, 44. \end{array}$$

90. **Use of the Derivative in Drawing Graphs.** We have seen (§ 76) that, if the equation of any curve is $y = f(x)$, the derivative of y with respect to x, which we have denoted by dy/dx, is also a function of x, unless $f(x)$ be linear in x. Let us denote this new function of x by $f'(x)$.

Thus (§ 76) if

$$y = f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

then

$$\frac{dy}{dx} = f'(x) = na_0x^{n-1} + (n-1)a_1x^{n-2} + (n-2)a_2x^{n-3} + \cdots + a_{n-1}.$$

Since the value of $f'(x)$ when $x = a$ is the slope (§ 75) of the curve $y = f(x)$ at the point whose abscissa is a, the expression for the derivative will be very useful in drawing the graph of a function $f(x)$. For if the abscissa of a point on the curve $y = f(x)$ is $x = a$, then the value of $f'(a)$ will tell us the direction in which the curve is going through this point. It is obvious that (§ 77)

(a) if $f'(a)$ is *positive*, the curve is *rising;*

(b) if $f'(a)$ is *negative*, the curve is *falling;*

(c) if $f'(a)$ is *zero*, the tangent to the curve at $x = a$ is *parallel to the x axis.*

EXAMPLE. Draw the graph of the function

$$y = f(x) = x^3 - x^2 - x + 2.$$

We have

$$f'(x) = 3x^2 - 2x - 1 = (3x + 1)(x - 1).$$

Let us now assign certain values to x and find the corresponding values of $f(x)$ and $f'(x)$.

x	-2	-1	$-\frac{1}{3}$	0	1	2
y	-8	1	$\frac{59}{27}$	2	1	4
$f'(x)$	15	4	0	-1	0	7

If we indicate at each point the direction of the tangent to the curve at that point, the graph is easily drawn (Fig. 27).

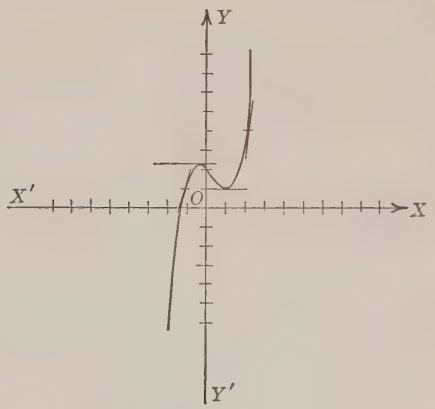


FIG. 27

In this example the roots of the equation $f'(x) = 0$ are readily determined and these values of x were included in the above table. They determine the *bending points* of the curve (§ 77).

91. Maximum and Minimum Values of a Polynomial. A *maximum value* of a function is a value greater than those immediately preceding or immediately following. A *minimum value* of a function is a value less than those immediately preceding or immediately following.

If the function is represented by the equation $y = f(x)$, the maximum ordinates M_1P_1, M_2P_2, M_3P_3 are greater than the ordinates immediately to the right or left of them, and the minimum ordinates N_1Q_1, N_2Q_2 are less than the ordinates immediately to the right or left of them. A maximum ordinate is not necessarily the greatest ordinate of the curve. In the figure the minimum ordinate N_2Q_2 is greater than the maximum ordinate M_1P_1.

It is evident that at a maximum or minimum point of a curve the tangent to the curve is parallel to the x axis. That is, $f'(x) = 0$ for a maximum or a minimum point.

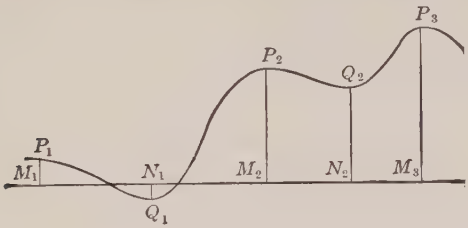


FIG. 28

It does not follow, conversely, that if $f'(x) = 0$ for a given point, the point is necessarily a maximum or a minimum point. The curve may cross the tangent at the given point. A point of a curve at which the curve crosses its tangent is called a ***point of inflexion.***

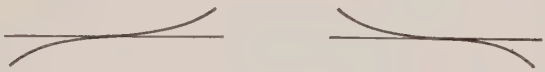


FIG. 29

In order to locate the maximum and minimum points of a curve we must find all the points of the curve for which $f'(x) = 0$, and then determine which of these are maximum points and which are minimum points, and which are points of inflexion.

A glance at Fig. 28 shows that if we think of a curve as generated by a point moving from left to right, the slope of the curve will change from positive to negative at a maximum point and from negative to positive at a minimum point.

The conditions for maximum and minimum points may now be stated as follows:

As the generating point passes through a point P of the curve for which $f'(x) = 0$,

P is a *maximum point*, if $f'(x)$ changes from $+$ to $-$,

P is a *minimum point*, if $f'(x)$ changes from $-$ to $+$,

P is a *point of inflexion*, if $f'(x)$ does not change sign.

EXAMPLE 1. Find the maximum and minimum values of the polynomial (§ 90)

$$f(x) = x^3 - x^2 - x + 2.$$

We have

$$f'(x) = (3x + 1)(x - 1).$$

The roots of the equation $f'(x) = 0$ are $x = -1/3$ and $x = 1$. Now $f'(x)$ is the product of two factors whose signs are

$(-)\ (-)$, when $x < -1/3$;
$(+)\ (-)$, when $-1/3 < x < 1$;
$(+)\ (+)$, when $x > 1$.

Thus the slope changes from positive to negative when $x = -1/3$ and then from negative to positive when $x = 1$. That is, $f(-1/3) = 59/27$ is a maximum value, and $f(1) = 1$ is a minimum value, of the function.

EXAMPLE 2. Find the maximum and minimum values of the polynomial

$$f(x) = x^3 - 3x^2 + 3x.$$

We have

$$f'(x) = 3(x^2 - 2x + 1) = 3(x - 1)^2.$$

Since $f'(x)$ is positive for all values of x, except $x = 1$, the curve has no maximum or minimum points. The point whose abscissa is $x = 1$ is a point of inflexion.

When $x =$	-1	$-\frac{1}{2}$	0	1	2	3
then $y =$	-7	$-\frac{19}{8}$	0	1	2	9
and $f'(x) =$	12	$\frac{27}{4}$	3	0	3	12

Fig. 30 shows the graph of this function.

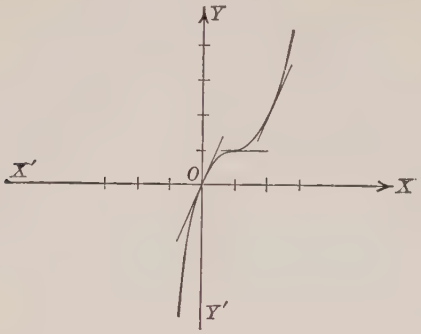


FIG. 30

EXERCISES

Draw the graphs of the following functions, determining in each case the maximum and minimum values, and also the zeros (§ 70), of the function.

1. $y = x^3 - 3x^2 + 4$.
2. $y = x^3 - 2x^2 + x$.
3. $y = x^4 - 8x^2 + 2$.
4. $y = 2x^3 - 11x^2 + 12x$.
5. $y = x^4 - 5x^2 + 4$.
6. $y = x^3$.
7. $y = x^3 - 8$.
8. $y = 3x^4 - 4x^3 + 1$.
9. $y = 6 - 5x - x^3$.
10. $y = 1 + 2x^2 - 3x^3$.

92. **Solution of Equations of Higher Degree.** In Chapters I and II methods are given for the solution of the general equations of the first and second degree. It is also possible to derive formulas for the solution of the cubic equation

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0,$$

and of the biquadratic equation

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0,$$

but the formulas are quite complicated and will not be considered here.

In this chapter we shall discuss methods for finding the exact value of each rational root, and the approximate value of each irrational root of the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0,$$

where a_0, a_1, a_2, $\cdots$, a_n are rational numbers and n is a positive integer.

93. **Fundamental Theorem.** We shall assume without proof the fundamental theorem of algebra which was first proved by Karl Friedrich Gauss (1777–1855).

Every algebraic equation has at least one root, real or imaginary.

By making use of this theorem it may be shown that every equation of the nth degree has exactly n roots.

94. **Location Principle.** It has already been pointed out that the abscissas of the points where the curve $y = f(x)$ crosses or touches the x-axis will be roots of the equation

$$f(x) = 0.$$

Thus in Fig. 26 we see that the roots of the equation

$$2x^4 + x^3 - 9x^2 - 4x + 4 = 0$$

are -2, -1, $\frac{1}{2}$, and 2. Also Fig. 27 shows that the equation $x^3 - x^2 - x + 2 = 0$ has only one real root, and that this root lies between -2 and -1.

95. **Theorem Regarding Rational Roots.** At this point we shall state and prove a theorem which will be very useful in finding the integral roots of any given equation.

THEOREM. *If $f(x)$ has integral coefficients and the coefficient of the highest power of x is unity, then any rational root of the equation $f(x) = 0$ is an integer and is a divisor of the constant term.*

PROOF. Let $f(x) = 0$ be written in the form

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0. \tag{1}$$

Suppose one root of this equation is the fraction p/q in its lowest terms. We then have

$$\left(\frac{p}{q}\right)^n + a_1\left(\frac{p}{q}\right)^{n-1} + a_2\left(\frac{p}{q}\right)^{n-2} + \cdots + a_{n-1}\left(\frac{p}{q}\right) + a_n = 0,$$

or, clearing of fractions by multiplying by q^n,

$$p^n + a_1p^{n-1}q + a_2p^{n-2}q^2 + \cdots + a_{n-1}pq^{n-1} + a_nq^n = 0. \tag{2}$$

Dividing by q, we find

$$\frac{p^n}{q} + a_1p^{n-1} + a_2p^{n-2}q + \cdots + a_{n-1}pq^{n-2} + a_nq^{n-1} = 0.$$

Since all the letters represent integers, every term after the first is obviously an integer. Hence the first term must be an integer; that is, q must be a factor of p^n. Since p and q have no common factors, this is impossible unless $q = 1$. Hence $p/q = p$, an integer. Thus every root of the equation is an integer. If $q = 1$, equation (2) reduces to

$$p^n + a_1p^{n-1} + a_2p^{n-2} + \cdots + a_{n-1}p + a_n = 0,$$

or, after dividing by p,

$$p^{n-1} + a_1p^{n-2} + a_2p^{n-3} + \cdots + a_{n-1} + \frac{a_n}{p} = 0.$$

Since all the letters represent integers, it is evident that p must be a factor of a_n, which completes the proof of the theorem.

By making use of the Remainder Theorem (§ 89) we may now quickly find all the integral roots of any equation of the type (1).

EXAMPLE. Solve the equation

$$x^3 - 6x^2 + 11x - 6 = 0.$$

The only possible integral roots are ± 1, ± 2, ± 3, ± 6. Let us now try each of these numbers in turn. The work in brief is as follows:

$$\begin{array}{rrrr|l} 1 & -6 & +11 & -6 & 1 \\ & +1 & -5 & +6 & \\ \hline 1 & -5 & +6 & +0 & \end{array}$$

Since the remainder is zero, one root of the equation is $x = 1$. It is also obvious that the quotient obtained by dividing the left member of the given equation by $x - 1$ is $x^2 - 5x + 6$. The other two roots of the given equation may now be found by solving the quadratic equation $x^2 - 5x + 6 = 0$, whose roots are $x = 2$ and $x = 3$.

Thus the roots of the given equation are $x = 1$, $x = 2$, and $x = 3$.

If the coefficient of the highest power of x is not unity, the equation may have fractional roots which may be rather difficult to find by the preceding method. In this case it is desirable to change the given equation into another equation whose roots are m times the roots of the given equation, choosing such a value for m that the roots of the new equation will be integers.

96. **Transformation of Equations.** Any process by which a given equation is changed into another equation whose roots are related to those of the given equation in a prescribed way is called ***transforming the equation.*** Examples of transformation of equations will be found in §§ 97 and 106.

The solution of an equation may frequently be simplified in this way.

97. **Multiplying the Roots by m.** To find the equation whose roots are m times the roots of the equation

$$(1) \qquad a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0,$$

we let $y = mx$; that is, we substitute y/m for x and obtain

$$\frac{a_0y^n}{m^n} + \frac{a_1y^{n-1}}{m^{n-1}} + \frac{a_2y^{n-2}}{m^{n-2}} + \cdots + \frac{a_{n-1}y}{m} + a_n = 0.$$

Clearing of fractions, we find

$$(2) \quad a_0y^n + ma_1y^{n-1} + m^2a_2y^{n-2} + \cdots + m^{n-1}a_{n-1}y + m^na_n = 0.$$

Since $y = mx$, it is evident that each of the n roots of (2) is m times the corresponding root of (1). If we denote the unknown in (2) by x instead of y, this equation becomes

$$(3) \quad a_0x^n + ma_1x^{n-1} + m^2a_2x^{n-2} + \cdots + m^{n-1}a_{n-1}x + m^na_n = 0.$$

Hence, *to multiply the roots of an equation by m, multiply the coefficients successively by m, m^2, $\cdots$, m^n, beginning with the second term.* Before applying this process to a given equation all missing terms must be supplied with zero coefficients.

Thus the equation whose roots are twice the roots of the equation $x^4 - 3x^3 + 4x - 5 = 0$ is $x^4 - 6x^3 + 32x - 80 = 0$.

It should be observed that the resulting equation is not a new form of the original equation. It is a different equation whose roots are related to those of the given equation in a certain definite way.

It is frequently desirable to transform an equation into another whose coefficients are integers and the coefficient of the highest power equal to unity. This may always be accomplished by multiplying the roots by some number m.

Thus the equation $3x^3 - x^2 + x - 4 = 0$ may be written in the form

$$x^3 - \frac{x^2}{3} + \frac{x}{3} - \frac{4}{3} = 0.$$

Then the equation whose roots are 3 times the roots of this equation is

$$x^3 - x^2 + 3x - 36 = 0.$$

98. **Changing the Signs of the Roots.** If we wish to change the signs of the roots of an equation, we must multiply each root by -1. That is, we take $m = -1$ in § 97. The multipliers $m, m^2, m^3, \cdots$, then become $-1, 1, -1, \cdots$.

Hence, *to change the signs of the roots of an equation, change the signs of all terms of odd degree, or of all terms of even degree.*

99. **Finding Rational Roots.** All integral roots of the given equation should first be found *by trial*, using the method of synthetic division. After each root has been found the linear factor corresponding to this root should be divided out, thus diminishing the degree of the equation by unity. This new equation is known as the ***depressed equation.***

After all integral roots have been found, the depressed equation should be reduced to the form (1) of § 95, by multiplying the roots by a suitably chosen value of m. The integral roots of this new equation may then be found by trial. In this way all the *rational* roots of the given equation will be obtained. Sometimes we can solve an equation in this way, as shown in the following example.

EXAMPLE. Find the rational roots of the equation

$$3x^5 + 13x^4 - 4x^3 - 37x^2 + 31x - 6 = 0. \tag{1}$$

The only possible integral roots are $\pm 1, \pm 2, \pm 3, \pm 6$. We may arrange our work as follows:

3	+13	− 4	−37	+31	− 6	1
	+ 3	+16	+12	− 25	+ 6	
3	+16	+12	− 25	+ 6	+ 0	
3	+16	+12	− 25	+ 6	−3	
	− 9	− 21	+ 27	− 6		
3	+ 7	− 9	+ 2	+ 0		

The only integral roots are $x = 1$ and $x = -3$, and the left member of equation (1) may be factored as follows:

$$(x-1)(3x^4+16x^3+12x^2-25x+6)$$
$$= (x-1)(x+3)(3x^3+7x^2-9x+2) = 0.$$

A simple trial shows that $3x^3+7x^2-9x+2$ cannot be divided again by $x-1$ nor by $x+3$.

We must now solve the depressed equation

$$3x^3+7x^2-9x+2=0.$$

Multiplying the roots by 3 and dividing by the coefficient of x^3, we find

$$x^3+7x^2-27x+18=0. \tag{2}$$

Since we have already eliminated all integral roots of (1), it is obviously useless to try any integers in (2) except those which are not multiples of 3. Thus the only possible integral roots of (2) are $\pm 1, \pm 2$. We find that $+2$ is a root.

1	+7	−27	+18	| 2
	+2	+18	−18	
1	+9	−9	+0	

The depressed equation is $x^2+9x-9=0$, which is a quadratic equation with roots, $x = (-9 \pm 3\sqrt{13})/2$. The roots of (2) are, therefore, $x = 2$ and $x = (-9 \pm 3\sqrt{13})/2$.

Dividing each of these roots by 3, we find as roots of equation (1), $x = 2/3$ and $x = (-3 \pm \sqrt{13})/2$. Thus the five roots of (1) are $x = 1$, $x = -3$, $x = 2/3$, $x = (-3 \pm \sqrt{13})/2$.

EXERCISES

Find the integral roots of the following equations. When possible find all the roots.

1. $x^3-8x^2+13x-6=0$.
2. $x^3-7x+6=0$.
3. $x^4-2x^3-3x^2+8x-4=0$.
4. $x^4-4x^3-8x+32=0$.
5. $x^4-6x^3+24x-16=0$.
6. $x^4-22x^2-12x+48=0$.
7. $x^4-6x^3+6x^2+5x+12=0$.
8. $x^4+x^3-22x^2-16x+96=0$.

Form new equations whose roots are equal to those of the following equations multiplied by the number indicated.

9. $x^3 - 2x^2 + 4x - 6 = 0$, 3. **11.** $x^3 - 4x^2 + 8x - 24 = 0$, $\frac{1}{2}$.

10. $2x^3 + x^2 - 4x + 3 = 0$, 2. **12.** $x^4 - x^2 + 3x + 7 = 0$, 3.

13. $4x^5 + 6x^3 - 2x + 1 = 0$, 2.

14. $2x^4 + 5x^3 - x^2 + 2x - 3 = 0$, -2.

Form new equations whose roots are m times the roots of the following equations, choosing values for m such that the new equations shall have integral coefficients, the coefficient of the first term being unity.

15. $2x^3 - 3x^2 - x + 4 = 0$. **17.** $3x^3 - 2x - 1 = 0$.

16. $6x^4 - 4x^3 + 3x - 8 = 0$. **18.** $4x^4 - 2x^3 + 3x^2 + 1 = 0$.

Form new equations whose roots are numerically equal but of opposite sign to the roots of the following equations.

19. $5x^3 + 2x^2 - x + 3 = 0$. **21.** $x^4 - 3x^3 + 4x^2 - 7x + 6 = 0$.

20. $2x^3 + 3x^2 + 6 = 0$. **22.** $3x^4 - 5x^3 - 2x^2 - 3x - 6 = 0$.

Find the rational roots of the following equations. When possible find all the roots.

23. $2x^3 + 9x^2 - 11x + 3 = 0$. **24.** $2x^3 - 5x^2 - 11x - 4 = 0$.

25. $8x^3 - 10x^2 + 12x - 15 = 0$.

26. $4x^4 - 8x^3 + 23x^2 - 40x + 15 = 0$.

27. $6x^4 - 5x^3 - 30x^2 + 20x + 24 = 0$.

28. $6x^4 - x^3 - 17x^2 + 16x - 4 = 0$.

29. $x^4 - 9x^3 + 17x^2 + 27x - 60 = 0$.

30. $2x^4 + 7x^3 + 9x^2 + 7x + 2 = 0$.

31. $3x^4 - 5x^3 - 17x^2 + 13x + 6 = 0$.

32. $9x^4 + 6x^3 - 59x^2 - 20x + 100 = 0$.

33. $x^5 - 3x^4 - 9x^3 + 21x^2 - 10x + 24 = 0$.

34. $x^5 + 8x^4 - 7x^3 - 103x^2 + 69x + 18 = 0$.

35. $x^5 - 3x^4 - 4x^3 + 26x^2 - 40x + 24 = 0$.

36. $x^6 - 7x^5 + 11x^4 - 7x^3 + 14x^2 - 28x + 40 = 0.$

37. $2x^6 + 11x^5 + 24x^4 + 22x^3 - 8x^2 - 33x - 18 = 0.$

38. $x^7 + x^6 - 14x^5 - 14x^4 + 49x^3 + 49x^2 - 36x - 36 = 0.$

100. **Irrational Roots.** By the foregoing methods it is always possible to find all of the rational roots of any given equation. After the factors which correspond to these roots have been removed, the roots of the depressed equation, if real, must be irrational. If, therefore, we are to find all the real roots of a given equation, we must find some way to determine the number of irrational roots of the equation, and then discover some method of finding the approximate values of these roots.

101. **Variations of Sign.** When the signs of two consecutive terms of a polynomial are unlike, a ***variation of sign*** is said to occur.

Thus, $x^5 + 2x^4 - 5x^3 - 4x^2 + x - 6$ has three variations of sign.

The following theorem will be needed in what follows.

THEOREM. *If a polynomial $f(x)$ with real coefficients has a factor $x - r$, where r is positive, the quotient of $f(x)$ divided by $x - r$ has fewer variations of sign than $f(x)$.*

Let us divide

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$$

by $x - r$, using synthetic division.

$$\begin{array}{l} a_0 + a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n \,\big\lfloor r \\ \quad + ra_0 + rA_1 + rA_2 + \cdots + rA_{n-2} - a_n \\ \hline a_0 + A_1 + A_2 + A_3 + \cdots + A_{n-1} \quad 0 \end{array}$$

The quotient is

$$a_0x^{n-1} + A_1x^{n-2} + A_2x^{n-3} + \cdots + A_{n-1},$$

where

$$A_1 = a_1 + ra_0,$$
$$A_2 = a_2 + rA_1,$$
$$A_3 = a_3 + rA_2, \quad \text{etc.}$$

Suppose a_3 is the first one of the coefficients $a_0, a_1, a_2, \cdots$, which is negative. Then, since r is positive, each of the coefficients a_0, A_1, and A_2 will be positive. The next coefficient, A_3, may be positive or negative. Hence a variation $+ -$ cannot occur in the quotient unless a similar variation occurs in the same part or an earlier part of $f(x)$.

Similarly, corresponding to each variation in the quotient, there is one or more like variations in $f(x)$. That is, there cannot be more variations of sign in the quotient than there are in $f(x)$.

Since r is positive, the sign of the last coefficient in the quotient is certainly opposite to that of the last coefficient in $f(x)$, for the remainder must be zero. Therefore $f(x)$ has at least one more variation than the quotient, and the theorem is proved.

Here is an example where A_4 is the first negative coefficient in the quotient, although a_1 and a_3 are both negative.

$$\begin{array}{r} 1 - 1 + 2 - 20 - 14 + 8 - 6 \\ \quad + 3 + 6 + 24 + 12 - 6 + 6 \\ \hline 1 + 2 + 8 + \ 4 - \ 2 + 2 \end{array} \Big\lfloor 3$$

102. **Descartes' Rule of Signs.** Suppose the equation $f(x) = 0$ has m positive roots $r_1, r_2, r_3, \cdots, r_m$. Divide $f(x)$ by $x - r_1$; then divide the quotient by $x - r_2$; and so on until all the factors corresponding to positive roots have been divided out. By § 101 each of m divisions which must be performed in order to remove all the positive roots of the equation will cause $f(x)$ to lose at least one variation of sign. Consequently the given function $f(x)$ must have at least m

variations of sign. This fact was first proved by the French mathematician Réné Descartes (1569–1650), and may be stated as follows:

THEOREM. *The equation $f(x) = 0$ with real coefficients cannot have more positive roots than $f(x)$ has variations of sign.*

Let us now replace x in $f(x)$ by $-x$. Since the roots of the new equation $f(-x) = 0$ are the same as those of $f(x) = 0$, but of opposite signs (§ 98), the positive roots of $f(-x) = 0$ are the negative roots of $f(x) = 0$.

Hence, by Descartes' Rule, *$f(x) = 0$ cannot have more negative roots than $f(-x)$ has variations of sign.*

103. **Use of Descartes' Rule of Signs.** The theorems of § 102 are very useful in the solution of higher equations. For example, if $f(x)$ has two variations of sign and we have already found two positive roots, we need not try to find another; and similarly for negative roots.

It should be observed, however, that Descartes' Rule does not give us the *exact* number of positive and negative roots of a given equation. It merely states an upper limit of the number of positive and negative roots.

Thus the left member of the equation $x^3 - x^2 - x + 2 = 0$ has two variations of sign. Hence we are sure that the number of positive roots does not exceed two. However, it is obvious from the graph (Fig. 27) that this equation has no positive roots.

By making use of Descartes' Rule we may sometimes discover the presence of imaginary roots of an equation.

Let us consider the equation

$$f(x) = x^5 + 7x^4 - 3x^2 - 5 = 0.$$

There is only one variation of sign in $f(x)$. Hence the equation cannot have more than one positive root. If we change x to $-x$, we obtain

$$f(-x) = -x^5 + 7x^4 - 3x^2 - 5,$$

which has only two variations of sign. Hence the equation $f(x) = 0$ cannot have more than two negative roots. Since the equation must have five roots, at least two of them must be imaginary.

EXERCISES

What may be said with reference to the number of positive, negative, and imaginary roots of the following equations?

1. $x^3 + 2x^2 - 4x - 2 = 0$.
2. $2x^3 - 4x^2 + x - 1 = 0$.
3. $x^4 + x^2 - 2x + 6 = 0$.
4. $3x^4 - x^3 + 2x^2 + 4 = 0$.
5. $x^5 - 4x^2 + 8 = 0$.
6. $x^7 - 3x^4 + 7x^3 - 5 = 0$.
7. $x^6 - 2x^5 + 3x^3 + 5x^2 + 4 = 0$.
8. $x^4 + 3x^3 + 2x^2 - 4x - 10 = 0$.

9. Show that the equation $x^6 + 3x^4 + 5x^2 + 12 = 0$ has no real root.

10. Show that the equation $2x^5 + 4x^3 - x^2 + 5 = 0$ has at least two imaginary roots.

11. Show that the equation $2x^7 - 8x^4 + 6x^3 - 3 = 0$ has at least four imaginary roots.

12. Show that the equation $x^3 + px + q = 0$, where p and q are positive, has one real negative root and two imaginary roots.

13. Show that the equation $x^n + 1 = 0$ has no real root when n is even; and but one real root when n is odd.

14. Show that an equation $f(x) = 0$ in which the terms are all positive and of even degree has no real root.

104. **Locating Irrational Roots.** We have seen (§ 99) that the rational roots of any equation may be quickly found and their corresponding factors removed from the equation. This should always be done before considering the irrational roots.

We must now discover some method of determining the approximate values of the irrational roots, which are, of course, roots of the depressed equation. If a table of values

of the function $f(x)$ corresponding to certain assigned values of x be constructed, the location of the irrational roots of the equation $f(x) = 0$ may usually be determined.

Consider the polynomial

$$f(x) = x^3 - 4x^2 + 2x + 2.$$

When $x =$	-2	-1	0	1	2	3	4	5
then $f(x) =$	-26	-5	2	1	-2	-1	10	37

The table of values shows that $f(x)$ changes sign between $x = -1$ and $x = 0$, between $x = 1$ and $x = 2$, and between $x = 3$ and $x = 4$. That is, the curve crosses the x axis between the points indicated. Hence the equation

$$x^3 - 4x^2 + 2x + 2 = 0$$

has three real roots (§ 94), which lie between -1 and 0, 1 and 2, and 3 and 4, respectively.

It should be observed that if any two values of x give values of $f(x)$ of opposite signs, we cannot conclude that there is only one root of the equation between these two values of x. There may be one, three, five, or any odd number of roots which may be separated by making the table more extensive. For example, the above function of x is negative for $x = -1$ and positive for $x = 4$, and there

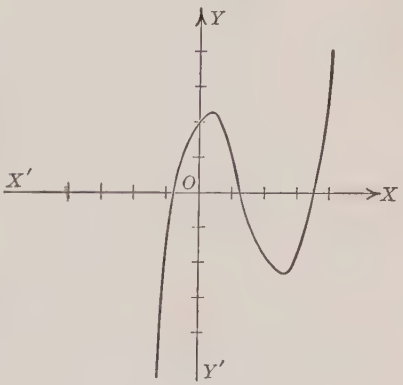


FIG. 31

are three roots of the equation between these numbers. Similarly if the function does not have opposite signs for two given values of x, we cannot conclude that there are no roots between the numbers. There may be an even number.

After locating the irrational roots we can then find an approximate value of each root in the following manner.

Let us consider the root which lies between 1 and 2. We infer from the above table of values that this root is nearer 1 than 2. Substituting 1.3 for x, we find, by the method of synthetic division, $f(1.3) = 0.037$; hence $x = 1.3$ is a closer approximation to the root than either $x = 1$ or $x = 2$.

x	$f(x)$
1	1
2	−2

The substitution of 1.4 for x gives $f(1.4) = -0.296$ Since $f(x)$ has opposite signs for $x = 1.3$ and $x = 1.4$, the root lies between 1.3 and 1.4 We may also infer that it is nearer to 1.3 We are now sure that the value of the root to one decimal place is 1.3

x	$f(x)$
1.3	0.037
1.4	−0.296

We may now carry the approximation one decimal place farther by substituting values for x between 1.3 and 1.4; a continuation of this process will give an approximation to any desired number of decimal places, but the calculations soon become very laborious. The remainder of this chapter will be devoted to a study of two methods of reducing this labor. It may be remarked, however, that the process already explained is sufficient to determine the roots to any desired degree of accuracy, and that the only reason for further work is to reduce the labor of it.

The abbreviated processes to be described will be applied only to equations of the form (1), page 184; and only when the coefficients are real numbers. The general process just described is often helpful in solving other types of equations, for example, simultaneous equations. See also § 108.

EXERCISES

Locate between consecutive integers the real roots of the following equations.

1. $x^3 + 5x - 7 = 0$.
2. $x^3 - 6x^2 + 8x + 40 = 0$.
3. $x^3 - 7x^2 - 3x + 8 = 0$.
4. $4x^3 - 2x^2 - x + 6 = 0$.
5. $20x^3 - 24x^2 + 3 = 0$.
6. $10x^3 - 17x^2 + x + 5 = 0$.
7. $2x^4 - 7x^3 - 4x + 1 = 0$.
8. $x^4 - 4x^3 + x^2 + 6x + 2 = 0$.
9. $x^4 - 10x^2 - 4x + 8 = 0$.
10. $2x^4 - 5x^3 + x - 3 = 0$.

Find to the nearest 0.1 one of the real roots of each of the following equations.

11. $x^3 + 13x^2 + 57x - 16 = 0$.
12. $x^3 - 15x - 5 = 0$.
13. $x^3 + 2x^2 + 5x + 5 = 0$.
14. $x^3 + x^2 - 7x - 3 = 0$.
15. $x^4 - 4x^3 - 3x + 23 = 0$.
16. $x^4 + 2x^3 - 6x^2 - 7x + 6 = 0$.

105. **Newton's Method of Approximation.** Let us suppose that $f(x)$ has opposite signs for $x = a$ and $x = b$ and that these two numbers are so close together that there is only one root between them. This is always possible unless the equation $f(x) = 0$ has a pair of equal roots, a case which we will not consider.

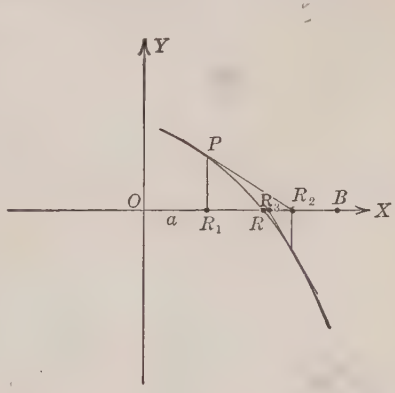


FIG. 32

Suppose the curve crosses the x axis at R between R_1 and B. Then the true value of the root which we are attempting to find is OR. If $OR_1 = a$, then $R_1P = f(a)$. Draw the tangent to the curve at P cutting the x axis at R_2. Then if $x = a$ is a good approximation to the

root, that is, if R_1 is close enough to R, the point R will be closer to R_2 than to R_1. Hence OR_2 will be a closer approximation to the root than OR_1.

Now the slope of the tangent at P (§ 90) is $f'(a)$. But by § 71 the slope of the line PR_2 is the quotient of the difference of the ordinates of P and R_2 divided by the difference of the corresponding abscissas. Hence

$$f'(a) = \frac{f(a)}{a - OR_2}.$$

Solving for OR_2, we find

$$OR_2 = a - \frac{f(a)}{f'(a)}.$$

Thus if $x = a$ is a good approximation to the root of the equation $f(x) = 0$, a better approximation will be

$$x = a - \frac{f(a)}{f'(a)}.$$

If we now draw the ordinate at R_2 and repeat the process, we will find that OR_3 is a better approximation, and so on.

If the curve crosses the axis from below, that is, if $f(x)$ changes from negative to positive at R, the figure will be different but the result will be the same.

Newton's method of approximation will in general fail if the derivative $f'(x)$ is zero, or the derivative of $f'(x)$, called the second derivative, is zero, near the point where the graph crosess the axis. The student should illustrate this fact graphically.

Example. Find the approximate value of the root between 1 and 2 of the equation

$$f(x) = x^3 - 4x^2 + 2x + 2.$$

We find by differentiation

$$f'(x) = 3x^2 - 8x + 2.$$

In § 104 we obtained as a first approximation $x = 1.3$; then a better approximation is

$$x = 1.3 - \frac{f(1.3)}{f'(1.3)} = 1.3 - \frac{0.037}{-3.33}$$
$$= 1.3 + .011 = 1.311$$

We can now get a closer approximation by starting with $a = 1.311$; then

$$x = 1.311 - \frac{f(1.311)}{f'(1.311)} = 1.311 - \frac{.0004}{-3.332}$$
$$= 1.311 + 0.0001 = 1.3111$$

Although we have carried the division to four decimal places, we cannot be sure of the result beyond the third decimal place. Thus the value of the root to three decimal places is $x = 1.311$ (compare § 107).

NOTE. If $f(a)$ should be negative, the point R_2 would be to the right of R.

EXERCISES

Find the root between:

1. 2 and 3 of $x^3 - 2x - 5 = 0$.

2. 0 and 1 of $x^3 - 7x + 3 = 0$.

3. -2 and -3 of $x^3 + 2x^2 - 5x - 7 = 0$.

4. 1 and 2 of $x^3 + x - 3 = 0$.

5. 3 and 4 of $x^3 - 2x^2 - 7x - 1 = 0$.

6. 2 and 3 of $x^4 - 6x^3 + 6x^2 + 8x + 1 = 0$.

106. **Diminishing the Roots by h.** When applying Newton's Method for approximating an irrational root of a given equation we think of the vector OR (Fig. 32), which is the graphical representation of the desired root, as the limiting value of the sequence of vectors OR_1, OR_2, OR_3, $\cdots$.

We shall now consider OR as the limiting value of another set of vectors, and in this way discover a second method for

finding the approximate value of an irrational root of an equation.

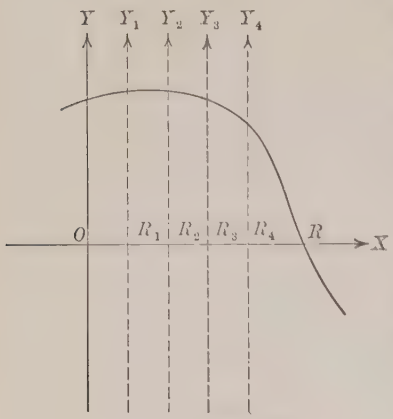


FIG. 33

Let the vector $OR_1 = h_1$ represent a first approximation to a root r of an equation $f(x) = 0$ (Fig. 33). Move the y axis parallel to itself until it passes through R_1.

The curve will evidently represent a different function of x when referred to the new set of axes. In fact the original equation will be transformed (§ 96) into a new equation whose roots will obviously be those of the given equation diminished by h_1. That is, if r_1 denotes the root of the transformed equation corresponding to r, then

$$r_1 = R_1R = r - h_1, \qquad \text{or} \qquad r = h_1 + r_1.$$

Let R_2 be some point between R_1 and R. Then $R_1R_2\ (= h_2)$ will be an approximation to the root r_1 of the transformed equation. Move the y axis again until it passes through R_2. Then, if r_2 is the corresponding root of the second equation, we shall have

$$r_2 = R_2R = r_1 - h_2, \qquad \text{or} \qquad r = h_1 + r_1 = h_1 + h_2 + r_2.$$

If this process be carried on indefinitely with suitably chosen values of h_1, h_2, $\cdots$, the y axis will, after several transformations, be very close to the point R and the root of the transformed equation will be very small. If, therefore, after several transformations, we assume that the root

of the transformed equation is zero, we obtain as an approximate value of the root of the given equation

$$r = h_1 + h_2 + h_3 + h_4 + \cdots.$$

Before applying this method of approximation we must first find some algebraic process which will transform a given equation into a new equation whose roots are the roots of the original equation diminished by h.

In order to diminish the roots of the equation

$$(1) \quad a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

by h, we let $y = x - h$; that is, we substitute $y + h$ for x and obtain

$$(2) \quad a_0(y+h)^n + a_1(y+h)^{n-1} + \cdots + a_{n-1}(y+h) + a_n = 0.$$

Since $y = x - h$, it is evident that each of the n roots of (2) is less by h than the corresponding root of (1).

Hence, *to diminish the roots of an equation by h, replace x by x + h.*

Thus, if the roots of the equation $x^2 - 7x + 10 = 0$ are 2 and 5, the equation whose roots are -2 and 1 is

$$(x+4)^2 - 7(x+4) + 10 = 0,$$

or

$$x^2 + x - 2 = 0.$$

The labor involved in expanding and collecting the terms in (2) may be avoided by making use of the following method.

Let us suppose that we have expanded and collected these terms and that the result is

$$(3) \quad a_0y^n + A_1y^{n-1} + A_2y^{n-2} + \cdots + A_{n-2}y^2 + A_{n-1}y + A_n = 0,$$

where the values of the constants A_1, A_2, $\cdots$, A_n depend upon h and the original coefficients a_0, a_1, a_2, $\cdots$, a_n. Our problem now is to find the values of the coefficients in (3).

Replace y by $x - h$, thus changing (3) back to (1). We obtain

$$a_0(x-h)^n + A_1(x-h)^{n-1} + A_2(x-h)^{n-2} + \cdots + A_{n-2}(x-h)^2 + A_{n-1}(x-h) + A_n = 0. \tag{4}$$

It is obvious that if the left member of (4) be divided by $x - h$ the remainder will be A_n and the quotient will be

$$a_0(x-h)^{n-1} + A_1(x-h)^{n-2} + A_2(x-h)^{n-3} + \cdots + A_{n-2}(x-h) + A_{n-1}. \tag{5}$$

Thus we may compute A_n by dividing (1) by $x - h$.

If the quotient (5) be divided by $x - h$, the remainder will be A_{n-1} and the quotient will be

$$a_0(x-h)^{n-2} + A_1(x-h)^{n-3} + A_2(x-h)^{n-4} + \cdots + A_{n-2},$$

and by continuing this process the rest of the coefficients may be found.

In applying this process the method of synthetic division should always be used.

The method of increasing the roots by h is the same as decreasing the roots by $-h$.

EXAMPLE. Find the equation whose roots are 2 less than those of the equation

$$2x^4 - 3x^3 - 12x^2 + 7x + 6 = 0.$$

The work in brief is as follows:

2	− 3	− 12	+ 7	+ 6	2
	+ 4	+ 2	− 20	− 26	
2	+ 1	− 10	− 13	**− 20**	
	+ 4	+ 10	+ 0		
2	+ 5	+ 0	**− 13**		
	+ 4	+ 18			
2	+ 9	**+ 18**			
	+ 4				
2	**+13**				

Hence the required equation is

$$2x^4 + 13x^3 + 18x^2 - 13x - 20 = 0.$$

It may be easily shown that the roots of the given equation are -2, $-\frac{1}{2}$, 1, 3. Then the roots of the transformed equation are -4, $-\frac{5}{2}$, -1, 1.

107. **Horner's Method of Approximation.** Let us now find one of the positive irrational roots of the equation

$$x^3 - 4x^2 + 2x + 2 = 0, \tag{1}$$

by the method suggested in § 106. This method was first published in 1819 by W. G. Horner, an English mathematician.

(1) *Locate the positive roots between consecutive integers.*

By the usual method of synthetic division we find that equation (1) has roots between -1 and 0, 1 and 2, 3 and 4. Let us calculate the root between 1 and 2 so that we may compare our results with § 105. We must then diminish the roots of (1) by $h_1 = 1$ (§ 106).

(2) *Diminish the roots by 1.*

1	-4	$+2$	$+2$	1
	$+1$	-3	-1	
1	-3	-1	$+1$	
	$+1$	-2		
1	-2	-3		
	$+1$			
1	-1			

Then the root of equation (1) is

$$r = h_1 + r_1 = 1 + r_1,$$

where r_1 is a root of the first transformed equation

$$x^3 - x^2 - 3x + 1 = 0 \tag{2}$$

lying between 0 and 1. Let us now locate r_1 between consecutive tenths.

(3) *Locate the root of (2) between consecutive tenths.*

Much labor may be avoided here by observing that since the value of x is small, the first two terms of equation (2) will be even smaller and may, with reasonable safety, be neglected in determining the first significant digit in the root. This leaves

$$-3x + 1 = 0,$$

or

$$x = 0.3;$$

we may now locate the root by substituting several of the numbers

$$0.0,\ 0.1,\ 0.2,\ \cdots,\ 0.9,$$

in the vicinity of 0.3

Let us try $x = 0.3$ and $x = 0.4$

1	− 1.0	− 3.00	+ 1.000	0.3
	+ 0.3	− 0.21	− 0.963	
1	− 0.7	− 3.21	+ **0.037**	
1	− 1.0	− 3.00	+ 1.000	0.4
	+ 0.4	− 0.24	− 1.296	
1	− 0.6	− 3.24	− **0.296**	

Since the left member of equation (2) is positive when $x = 0.3$ and negative when $x = 0.4$, the root lies between these two numbers. Hence we must diminish the roots of (2) by $h_2 = 0.3$

It should be observed that since the approximations

$$h_1,\ h_2,\ h_3,\ \cdots,$$

are all on the same side of the root (Fig. 33) the sign of the last term of the successive transformed equations should be the same. If this term changes sign, it means that we are diminishing the roots by a number larger than the root, that is, we are moving the y axis beyond R (Fig. 33).

(4) *Diminish the roots of (2) by 0.3*

1	− 1.0	− 3.00	+ 1.000	0.3
	+ 0.3	− 0.21	− 0.963	
1	− 0.7	− 3.21	+ 0.037	
	+ 0.3	− 0.12		
1	− 0.4	− 3.33		
	+ 0.3			
1	− 0.1			

Then the root of equation (1) is

$$r = h_1 + h_2 + r_2 = 1.0 + 0.3 + r_2$$
$$= 1.3 + r_2,$$

where r_2 is a root of the second transformed equation

(3) $$x^3 - 0.1x^2 - 3.33x + 0.037 = 0$$

lying between 0.0 and 0.1

(5) *Locate the root of (3) between consecutive hundredths.*

Again neglecting the first two terms of (3), the equation $-3.33x + 0.037 = 0$ suggests $x = 0.01$ as an approximate value of the root of (3). Hence we must diminish the roots of (3) by $h_3 = 0.01$

The method of approximating the root by using only the last two terms of the equation will usually give the correct value when x is less than 0.1 However an error here will be quickly discovered. For if the value $x = 0.01$ happens to be too large the last term will change sign, and if too small, the root of the transformed equation will not be within the proper limits.

(6) *Diminish the roots of (3) by 0.01*

1	− 0.10	− 3.3300	+ 0.037000	0.01
	+ 0.01	− 0.0009	− 0.033309	
1	− 0.09	− 3.3309	+ 0.003691	
	+ 0.01	− 0.0008		
1	− 0.08	− 3.3317		
	+ 0.01			
1	− 0.07			

Then the root of equation (1) is

$$\begin{aligned} r &= h_1 + h_2 + h_3 + r_3 \\ &= 1.00 + 0.30 + 0.01 + r_3 \\ &= 1.31 + r_3, \end{aligned}$$

where r_3 is a root of the third transformed equation

$$(4) \qquad x^3 - 0.07x^2 - 3.3317x + 0.003691 = 0$$

lying between 0.00 and 0.01

(7) *Locate the root of (4) between consecutive thousandths.*

Neglecting the first two terms of (4) we find $x = 0.001$ Hence $h_4 = 0.001$ and the root of equation (1) is

$$\begin{aligned} r &= h_1 + h_2 + h_3 + h_4 + r_4 \\ &= 1.000 + 0.300 + 0.010 + 0.001 + r_4 \\ &= 1.311 + r_4, \end{aligned}$$

where r_4 lies between 0 and 0.001

Thus the required value of r to three places of decimals is $r = 1.311$

This process may be continued indefinitely and the value of r obtained, one digit at a time, correct to any number of digits. Newton's Method (§ 105) is much quicker if only one or two places of decimals are required.

In practice the preceding work is generally arranged in the following compact form:

1	− 4	+ 2	+ 2	1
	+ 1	− 3	− 1	
1	− 3	− 1	+ 1	
	+ 1	− 2		
1	− 2	− 3		
	+ 1			

1	− 1	− 3	+ 1	| .3
	+ 0.3	− 0.21	− 0.963	
1	− 0.7	− 3.21	+ 0.037	
	+ 0.3	− 0.12		
1	− 0.4	− 3.33		
	+ 0.3			

1	− 0.1	− 3.33	+ 0.037	| .01
	+ 0.01	− 0.0009	− 0.033309	
1	− 0.09	− 3.3309	+ 0.003691	
	+ 0.01	− 0.0008		
1	− 0.08	− 3.3317		
	+ 0.01			

1	− 0.07	− 3.3317	+ 0.003691	| .001

In order to determine a negative irrational root of an equation $f(x) = 0$, change x to $-x$ and determine the corresponding positive root of $f(-x) = 0$.

If the equation has a pair of equal roots, the above method must be somewhat modified. This case will not be considered here.

EXERCISES

Find the root between

1. 3 and 4 of $x^3 - 6x^2 + 15x - 19 = 0$.
2. 2 and 3 of $x^3 - 9x + 5 = 0$.
3. -2 and -3 of $x^3 - 7x + 3 = 0$.
4. 2 and 3 of $x^3 + 10x^2 + 8x - 120 = 0$.
5. -3 and -4 of $x^3 + 6x^2 + 9x + 1 = 0$.
6. 1 and 2 of $x^4 + 2x^3 - 5 = 0$.
7. -1 and -2 of $x^4 - 2x^3 - 3x^2 + x - 2 = 0$.
8. 1 and 2 of $x^3 - 3x^2 - 2x + 5 = 0$.
9. 3 and 4 of $2x^3 + x^2 - 15x - 59 = 0$.

Find to three places of decimals, the roots of

10. $x^3 - x^2 - 15x + 28 = 0$, between 2 and 3.

11. $x^4 - 6x^3 + 6x^2 + 8x + 1 = 0$, between 0 and -1.

12. The edges of a rectangular box are 4, 6, and 8 inches. The same amount is added to each dimension and the volume is doubled. Find to two decimal places the amount added.

13. A spherical shell has an inner radius of 10 inches. The solid part of the shell contains 2000 cu. in. of metal. Find the thickness of the shell.

14. If r is the radius of a sphere made from a substance whose specific gravity is s, it is known that the depth x to which the sphere will sink is a positive root of the equation, .

$$x^3 - 3rx + 4r^3s = 0.$$

Find x correct to two decimal places, if $r = 3$ inches and $s = .6$

15. In the equation of Ex. 14, find x, if $r = 2$ and $s = .3$

108. **Complex Roots.** Although we shall not attempt in this chapter to find the imaginary roots of an equation, we shall derive one important theorem with reference to complex roots.

THEOREM. *If a complex number $a + ib$ is a root of an equation with real coefficients, the conjugate complex number $a - ib$ is also a root.* That is, *complex roots always occur in pairs.*

We have seen that this theorem is true in the case of a quadratic equation (§ 20). Let us now prove it for the equation

$$(1) \quad f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0,$$

where a_0, a_1, a_2, $\cdots$, a_n are real.

Substituting $a + ib$ for x in $f(x)$ gives

$$(2) \quad a_0(a + ib)^n + a_1(a + ib)^{n-1} + \cdots + a_{n-1}(a + ib) + a_n.$$

If each of the parentheses be expanded by the Binomial Theorem, every even power of ib will give rise to a real term (since a and b are real), and every odd power of ib will give rise to a pure imaginary term. Then if P denotes the sum of the real terms and iQ denotes the sum of the pure imaginary terms, we have, since $a + ib$ is a root of (1),

$$P + iQ = 0. \tag{3}$$

Since P and Q are real, it follows from (3) that $P = 0$ and $Q = 0$.

If we now substitute $a - ib$ for x in (1), we will obtain equation (2) with b changed to $-b$. When the parentheses are expanded, the even powers of ib will be the same as before but every term involving an odd power of ib will have its sign changed. The result may, therefore, be written in the form $P - iQ$.

Since $P = 0$ and $Q = 0$, it follows that $P - iQ = 0$. That is, the substitution of $a - ib$ for x in (1) reduces $f(x)$ to zero. Thus $a - ib$ is a root of the equation $f(x) = 0$, which proves the theorem.

It is evident that if $a + ib$ is a root of the equation $f(x) = 0$, both of the expressions $x - (a + ib)$ and $x - (a - ib)$ are factors of $f(x)$. Consequently it is always possible to divide $f(x)$ by the product of these factors, $(x - a)^2 + b^2$, which is real. Thus if $f(x)$ has real coefficients and one imaginary root of the equation $f(x) = 0$ is known, the degree of the equation can be reduced by 2.

While we shall not take up the question in detail, it may be remarked that the general process of § 104 can be extended to find approximately the solutions of the simultaneous equations $P = 0$ and $Q = 0$. The only bar to finding the roots to any desired degree of accuracy is the labor involved.

CHAPTER XII

PERMUTATIONS AND COMBINATIONS

109. **Fundamental Theorem.** Suppose there are 4 motor busses operating between two towns A and B, 5 between B and C, and 2 between C and D. Let us investigate the number of ways in which a man can make the trip from A to D (via B and C). The trip from A to B may be made in any one of 4 busses and, having arrived at B, the trip to C may be made in any one of 5 busses. Since the choice of a bus for the second part of the trip is entirely independent of the choice in the first case, the trip from A to C (via B) may be made in 4×5, or 20 different ways. After arriving at C, the trip to D may be made in either of two busses. Hence the trip from A to D may be made in $4 \times 5 \times 2$, or 40 different ways.

Again, suppose a word of two letters is to be formed from the 21 consonants and 5 vowels of the English alphabet, the only condition being that the first letter shall be a consonant and the second a vowel. A consonant may be chosen in 21 different ways and to each of these 21 choices there will correspond 5 choices of a vowel. Hence the total number of words of two letters which satisfy the given condition is 21×5, or 105.

We are thus led to the following general theorem.

THEOREM. *If one thing can be done in n_1 different ways, and, when it is done, a second thing can be done in n_2 different ways, and, having done it, a third thing can be done in n_3 different ways, and so on, then the number of ways in which the various things can be done jointly is*

$$N = n_1 \cdot n_2 \cdot n_3 \cdots.$$

EXERCISES

1. A room has four doors. In how many ways can a person enter and leave by a different door?

2. A town has five hotels. Three people wish to stay at different hotels. In how many ways can this be done?

3. A man has three coats, four vests, and five pairs of trousers. In how many ways can he dress?

4. In how many ways can three prizes be awarded to ten students if any student may receive one, two, or three prizes?

5. How many Greek letter fraternities can be named from the twenty-four letters of the Greek alphabet, using either two or three letters at a time?

6. If the first call letter of a broadcasting station must be either K or W, how many different "calls" of four letters are possible? Of three letters?

7. How many numbers of five figures, having 3 as the middle digit, can be formed from the digits 1, 2, 3, 4, 5?

8. How many even numbers of five different figures each can be formed from the digits 1, 2, 3, 4, 5?

9. In how many different ways can a coach play nine men on a baseball team if either A, B, or C must pitch? If A must pitch and B must catch?

10. How many different signals can be given with flags of 5 different colors, using 1, 2, 3, 4, or 5 flags at a time?

11. How many odd numbers of five different figures each can be formed from the digits 5, 6, 7, 8, 9?

12. If the first two call letters of a broadcasting station must be K T, how many different "calls" of four letters are possible? Of three letters?

110. **Permutations.** The ***permutations*** of any set of objects are the different orders in which they can be arranged.

Thus there are six different permutations of the digits 2, 5, 8, as follows: 258, 285, 528, 582, 825, 852.

111. **Permutations of n Things Taken r at a Time.** Suppose we wish to know how many numbers of two digits each can be formed from the four digits 2, 5, 8, 9, if no digit is used more than once in any number. The first digit may evidently be chosen in any one of 4 ways and, after this choice has been made, the second digit may be chosen in 3 ways. Hence the total number of permutations of the 4 digits taken two at a time is 4×3, or 12. These permutations are

25, 52, 28, 82, 29, 92, 58, 85, 59, 95, 89, 98.

Let us now suppose that we have n different things to choose from and that we wish to know the number of permutations of these things taken r at a time. This is equivalent to filling r places with r things chosen from n things. The first place may be filled in any one of n different ways and after this choice has been made any one of the remaining $n - 1$ things may be chosen to fill the second place. Thus the second place may be filled in $n - 1$ different ways.

After the second place has been filled the third place may be filled in $n - 2$ different ways, and so on, until $r - 1$ places are filled. The last place may be filled in $n - (r - 1)$, or $n - r + 1$ ways.

Then *if the symbol* ${}_nP_r$ *denotes the number of permutations of n things taken r at a time, we have*

$$\boldsymbol{{}_nP_r = n(n-1)(n-2)\cdots(n-r+1).}$$

Thus the number of permutations of 6 things taken 4 at a time is ${}_6P_4 = 6 \times 5 \times 4 \times 3 = 360$. It should be observed that the number of factors in the right member of the above equation is the same as the number of things taken at a time.

112. **Permutations of n Things Taken All at a Time.** When $r = n$ the above formula reduces to

$${}_nP_n = n(n-1)(n-2)\cdots 2\cdot 1.$$

Since this is the product of all integers from n down to 1, which we have called factorial n (§ 53), we may say that *the number of permutations of n things taken all at a time is given by the formula*

$$_nP_n = n!$$

Example 1. How many even numbers greater than 40,000 can be formed from the digits 1, 3, 4, 5, 6, it being understood that each digit is to be used once and only once.

The first digit must be either 4, 5, or 6. If 4 be chosen for the first digit, then the last digit must be 6, and the other three digits can be arranged in 3! ways. Hence there will be 6 possible numbers beginning with the figure 4.

In like manner it may be shown that there will be 6 possible numbers beginning with the figure 6.

If 5 be chosen for the first digit, then the last digit must be either 4 or 6. Hence there will be $2 \times 3!$, or 12 possible numbers beginning with the figure 5.

Thus the total number of possible numbers will be $6 + 6 + 12 = 24$.

Example 2. In how many ways can A, B, C, D, and E be seated in a row if A and B cannot sit next to each other?

Consider A and B together as one element to be arranged with the other three elements. There will be 4!, or 24 arrangements of these 4 elements. But in each of these arrangements A and B may sit in the order AB or BA. Thus the total number of arrangements in which A and B sit next to each other is 2×24, or 48.

Now the total possible number of arrangements of the five men is 5! or 120. Hence there will be $120 - 48$, or 72 arrangements in which A and B will not sit next to each other.

113. **Permutations of Things Not All Different.** The number of permutations of the eight different letters a_1, a_2, a_3, b_1, b_2, c, d, e is 8! Let us now suppose that a_1, a_2, and a_3 are each replaced by a. Then each of the 3! permutations of the letters a_1, a_2, a_3 is replaced by the arrangement aaa.

Thus the total number of permutations of the letters a, a, a, b_1, b_2, c, d, e is $\frac{8!}{3!}$. If now b_1, b_2 be replaced by b, this number is reduced to $\frac{8!}{3!\,2!}$.

In general, *the number of permutations of n things, of which n_1 are alike, n_2 others are alike, n_3 others are alike, $\cdots$, is*

$$N = \frac{n!}{n_1!n_2!n_3!\cdots}.$$

EXAMPLE. In how many different ways can the letters of the word Mississippi be arranged?

The word contains four i's, four s's, and two p's. Hence the total number of different arrangements is

$$N = \frac{11!}{4!\,4!\,2!} = \frac{11\cdot 10\cdot 9\cdot \cancel{8}\cdot 7\cdot \cancel{6}\cdot 5\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1}{\cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1\cdot \cancel{2}\cdot 1} = 34{,}650.$$

EXERCISES

Find the value of

1. ${}_9P_4$. **2.** ${}_6P_6$. **3.** ${}_{14}P_6$. **4.** ${}_{30}P_4$.

Find the value of n when

5. ${}_nP_6 = 20\,{}_nP_4$. **6.** ${}_{2n+1}P_4 = 140\,{}_nP_3$.

7. Given ${}_nP_3 : {}_{n+2}P_3 = 5 : 12$, find n.

8. Show that $n! = n \times (n-1)!$.

9. In a certain play there are four parts to be taken by a company of ten persons. In how many different ways can they be assigned?

10. Four persons enter a room in which there are six chairs. In how many different ways can they be seated?

11. How many numbers can be formed by using six out of the nine digits 1, 2, 3, $\cdots$, 9?

12. How many different numbers can be made with the digits 5, 6, 7, 8, if repetitions are not allowed? If repetitions are allowed?

13. There are eight candidates for a basketball team, seven of whom can play in any position. If the other man can play only at center, in how many different ways can the coach play five men?

14. How many different words of seven letters each can be formed from seven letters, if three of the letters cannot be separated? How many if these three can be in only one order?

15. In how many different orders can the letters of the following words be written: (*a*) *Arkansas*, (*b*) *Philadelphia*, (*c*) *Massachusetts*, (*d*) *Engineering?*

16. In how many ways can fifteen balls be placed in a row, if six are black, five are red, and four are white?

17. How many different signals can be made with seven flags of which two are blue, three red, and two white, if all are hoisted for each signal?

18. How many numbers of seven figures can be formed from the digits 1, 2, 2, 3, 3, 3, 4?

19. In how many ways can seven men be arranged in a row if one particular man is not permitted to stand at either end?

20. In how many ways can six people be seated at a round table?

21. In how many ways can five men, two women, and two children be seated in a row so that the men sit together, the women sit together, and the children sit together?

22. Solve Ex. 21 when it is required that a man shall sit at each end of the row and the other seven people may be arranged in any manner whatever.

23. In how many ways can eight persons be seated in a row if A must have B on his right and C on his left? If A must sit between B and C?

24. How many whole numbers less than 400 and containing no repetition of digits can be formed from the digits 1, 2, 3, and 4?

25. Of the permutations possible with the letters of the word *associate*, taken all together, how many begin with a vowel?

26. In how many ways can seven ladies and seven gentlemen be seated at a round table so that no two ladies sit together?

27. In how many ways can eight people be seated in a row so that a certain two people will not sit together?

114. **Combinations.** The *combinations* of any set of objects are the different collections which may be formed from them without regard to the order in which they are placed.

Thus, if we do not consider the order in which the letters are written, *abc*, *acb*, *bac*, *bca*, *cab*, and *cba* are the same combination of letters although they represent six different permutations.

115. **Combinations of n Things Taken r at a Time.** We have seen (§ 111) that the total number of permutations of n different things taken r at a time is

$$_nP_r = n(n-1)(n-2)\cdots(n-r+1).$$

But each combination of r different things will have $r!$ permutations. Hence, if the order is not taken into account, *the number of combinations of n different things taken r at a time, which is generally denoted by $_nC_r$, is equal to the number of permutations divided by $r!$.* Thus

$$_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}.$$

Thus the number of combinations of 8 different things taken 4 at a time is

$$_8C_4 = \frac{\overset{2}{\cancel{8}}\cdot 7\cdot\cancel{6}\cdot 5}{1\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4}} = 70.$$

Note. Observe that the coefficient of the $(r+1)$th term (§ 13) of the binomial expansion for $(a+x)^n$ is $_nC_r$. Hence the Binomial Theorem for positive integral exponents may be written in the form

$$(a+x)^n = a^n + {}_nC_1a^{n-1}x + {}_nC_2a^{n-2}x^2 + \cdots + {}_nC_{n-1}ax^{n-1} + {}_nC_nx^n.$$

Thus

$$\begin{aligned}(a+x)^5 &= a^5 + {}_5C_1a^4x + {}_5C_2a^3x^2 + {}_5C_3a^2x^3 + {}_5C_4ax^4 + {}_5C_5x^5\\ &= a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5.\end{aligned}$$

EXAMPLE 1. Find the number of combinations of 20 different things taken 5 at a time.

$$_{20}C_5 = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 15{,}504.$$

EXAMPLE 2. Find the number of ways in which twelve persons can be divided into two committees of seven and five persons, respectively.

The number required is $_{12}C_7$, since every time a group of seven persons is selected a group of five persons is left.

EXAMPLE 3. How many different words, each containing three consonants and three vowels, can be formed from seven consonants and five vowels?

The number of combinations of the seven consonants, taken three at a time, is

$$_7C_3 = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 35.$$

The number of combinations of the five vowels, taken three at a time, is

$$_5C_3 = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 10.$$

Since any one of the thirty-five sets of consonants may be associated with any one of the ten sets of vowels, there will be 35×10, or 350 sets, each containing three consonants and three vowels.

Now each set of 6 letters can be arranged in 6!, or 720 different ways. Hence the total number of different words is 350×720, or 252,000.

EXERCISES

Find the value of

1. $_{10}C_4$. **2.** $_{12}C_9$. **3.** $_{15}C_6$. **4.** $_{12}C_{11}$.

Show that

5. $_nC_r = {}_nC_{n-r}$. **6.** $_nC_r = \dfrac{n!}{r!(n-r)!}$.

7. If $_{2n}C_{n-1} : {}_{2n-2}C_n = 132 : 35$, find the value of n.

8. How many straight lines can be drawn through ten points, no three of which are collinear?

9. How many diagonals has a polygon of twelve sides?

10. At a meeting of eight men, each shakes hands with all the others. How many handshakes are there?

11. A man has twenty friends, of whom twelve are men. In how many ways can he invite fourteen guests from them so that eight may be men?

12. From four men, six women, and ten children, how many different groups can be formed, each containing two men, two women and five children?

13. A basket contains five peaches, seven apples and six oranges. How many selections of five each can be made so as to contain at least one apple?

14. How many different words of six letters each can be formed of the twentysix letters of the alphabet, if each word must contain two vowels?

15. From twelve French and ten German books, all different, eight French and six German books are to be selected and are then to be arranged on a shelf. In how many ways can this be done?

16. A detachment of twenty men must furnish each night a guard of four men. For how many nights could different guards be formed from among the twenty men? How many times would each man serve?

17. A bag contains five red balls, four white balls, and three blue balls. In how many ways can a selection of three balls be made so that (1) all three are red, (2) none are red, (3) at least one is red?

18. From three capitals, five other consonants, and four other vowels, how many words can be made, each beginning with a capital and containing in addition three consonants and two vowels?

19. From seven sophomores and four freshmen a committee of six is to be formed. In how many ways can this be done, (1) when the committee contains exactly two freshmen, (2) at least two freshmen?

Show that

20. ${}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r$.

21. ${}_nC_1 + {}_nC_2 + {}_nC_3 + \cdots + {}_nC_{n-1} + {}_nC_n = 2^n - 1$.

22. In how many ways can a man select some fruit from a basket which contains an apple, an orange, a lemon, a peach, a plum, and a banana?

23. From a basket containing three red balls, four white balls and two blue balls, how many selections can be made, taking at least one of each kind?

24. From six men and eight women a committee of six is to be formed; in how many ways can this be done when the committee contains, (1) exactly two men, (2) not more than two men?

25. In how many ways can a person invite one or more of seven friends to dinner?

26. A coach has twelve men from whom to select a baseball team. If each man can play in any position, but A will not play unless B plays, in how many different ways can he play nine men?

27. A city council consists of a mayor and eight aldermen. How many different committees of four can be formed, (1) if the mayor must be included, (2) if the mayor may or may not be included, (3) if two of the members refuse to serve with the mayor?

28. In how many ways can eighteen marbles be divided equally among three boys?

29. A class consists of thirty members, ten of whom are girls. In how many ways can a committee of six be selected so as to include at least one girl?

CHAPTER XIII

PROBABILITY

116. Probability of a Single Event. If an event can happen in h different ways and fail in f ways, and all these ways are equally likely to occur, the *probability*, or the *chance*, of its happening is $p = h/(h + f)$, and the probability of its failing is $q = f/(h + f)$.

Thus if a ball is to be drawn from a bag containing four red balls and three white balls, a red ball may be drawn in four different ways and a red ball may fail to be drawn (by drawing a white ball) in three different ways. Hence the probability of drawing a red ball is $4/(4 + 3)$, or $4/7$.

Since $h/(h + f) + f/(h + f) = 1$, it is obvious that $p + q = 1$, or $q = 1 - p$.

Thus, *if p is the probability that an event will happen, the probability that it will fail is $1 - p$.*

If an event is sure to happen, then $q = 0$ and $p = 1$.

Instead of speaking of the probability of an event we sometimes say *the odds are p to q in favor of the event, or q to p against it.*

EXAMPLE 1. A bag contains tickets numbered 1, 2, 3, $\cdots$, 20, and two blanks.

(*a*) If five tickets are drawn, what is the probability that two of them are blanks?

If both blanks are to be drawn, we need consider only the number of combinations of the remaining twenty tickets taken three at a time. Three tickets may be drawn from 20 in ${}_{20}C_3$ different ways, and five tickets may be drawn from 22 in ${}_{22}C_5$ different ways.

Hence the probability that both blanks are drawn is

$$\frac{{}_{20}C_3}{{}_{22}C_5} = \frac{20 \cdot 19 \cdot 18}{1 \cdot 2 \cdot 3} \div \frac{22 \cdot 21 \cdot 20 \cdot 19 \cdot 18}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{4 \cdot 5}{22 \cdot 21} = \frac{10}{231}.$$

(*b*) If five tickets are drawn, what is the probability that neither of them is a blank?

The number of combinations of the twenty numbered tickets taken five at a time is ${}_{20}C_5$. The number of combinations of the twenty-two tickets taken five at a time is ${}_{22}C_5$. Hence the probability of drawing five numbered tickets is

$$\frac{{}_{20}C_5}{{}_{22}C_5} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \div \frac{22 \cdot 21 \cdot 20 \cdot 19 \cdot 18}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{17 \cdot 16}{22 \cdot 21} = \frac{136}{231}.$$

EXAMPLE 2. A bag contains five red, four white, and three black balls. A man is entitled to draw six balls and is to receive $20.00 in case he draws two balls of each color. What is the value of his expectation? *

The number of different ways of drawing (1) two red balls is ${}_5C_2$, or 10, (2) two white balls is ${}_4C_2$, or 6, (3) two black balls is ${}_3C_2$, or 3. Since we may associate any one of these combinations with any other, there are in all $10 \times 6 \times 3$, or 180 different combinations, each consisting of two balls of each color.

The number of combinations of the twelve balls taken six at a time is ${}_{12}C_6$, or 924.

Hence his chance of winning is 180/924, or 15/77; and the value of his expectation is $(15/77) \times \$20$, or $3.90.

EXAMPLE 3. What is the chance of throwing 7 in a single throw with two dice?

The event can happen in three ways: by throwing either 6 and 1, 5 and 2, or 4 and 3.

Each die can be thrown in six ways; and hence the pair can be thrown in 6×6, or 36 ways.

Now 6 and 1 can be thrown in two ways. Hence the chance of making this throw is 2/36, or 1/18. Likewise the chance of throwing 5 and 2 is 1/18, and the chance of throwing 4 and 3 is 1/18. Hence the chance of throwing 7 is $1/18 + 1/18 + 1/18$, or 1/6.

* If a man's chance of winning a sum of money S is p, then the value of his *expectation* is pS.

NOTE. If an event can happen in more than one way, and these ways are mutually exclusive, the chance of the happening of the event is the sum of the chances of its happening in the separate ways.

EXERCISES

1. What is the probability that two cards selected at random from a pack of fifty-two cards will both be aces?

2. If two balls are chosen at random from a box containing four white balls and three red balls, what is the probability of drawing

(*a*) two white balls?
(*b*) two red balls?
(*c*) one white and one red ball?

3. If three balls are drawn at random from a bag containing six red, four white, and five black balls, what is the probability that

(*a*) all are red?
(*b*) all are white?
(*c*) all are black?
(*d*) 2 red and 1 white?
(*e*) 1 red and 2 black?
(*f*) 1 white and 2 black?

4. A bag contains tickets numbered 1, 2, 3, $\cdots$, 12. If three tickets are drawn, what is the probability that

(*a*) they are 4, 8, and 12?
(*b*) either 4, 8, or 12 is among them?

5. If three coins be tossed, what is the probability that

(*a*) all will be heads?
(*b*) there will be exactly two heads?
(*c*) there will be at least two heads?

6. What is the probability of throwing at least 6 in a single throw with two dice?

7. In one throw with a pair of dice what number is most likely to be thrown? What are the odds against throwing that number?

8. What is the probability of throwing 13 in a single throw with three dice?

9. What is the probability of throwing at least one ace in a single throw with three dice?

10. If eight men stand in a line, what is the chance that A and B are next to each other?

11. From a committee of eight Republicans and six Democrats a sub-committee of three is chosen by lot. What is the probability that it will contain more Republicans than Democrats?

12. A bag contains six five-dollar bills and eight one-dollar bills. If a person is entitled to draw two bills, what is the value of his expectation?

13. Nine persons are seated at random at a round table; what is the probability that A and B will be seated together? That C will be seated between A and B?

117. **Probability of Several Events.** Up to this point we have confined our attention to the probability of the happening of a single event under different conditions. If we wish to apply the theory of probability to two or more events, we must distinguish between those events which are independent and those which are dependent.

Two or more events are said to be *independent* if the happening of one of them does not affect the happening of the others. If two or more events are not independent, they are said to be *dependent.*

Suppose a bag contains two white and two black balls and a man is entitled to draw two balls (one after the other). If the ball is replaced after the first drawing, the events are independent and his chance of drawing a white ball on the second trial is $\frac{1}{2}$; but if the ball is not replaced, his chance depends on the color of the ball which was drawn first.

118. **Independent Events.** Let us consider two independent events, which can happen in h_1 and h_2 different ways and fail in f_1 and f_2 different ways, respectively, where all these ways are equally likely to occur.

The first event can happen, or fail, in $h_1 + f_1$ ways, and the second in $h_2 + f_2$ ways, and any one of these cases may be associated with any one of the others, hence there are $(h_1 + f_1)(h_2 + f_2)$, or $h_1h_2 + h_1f_2 + f_1h_2 + f_1f_2$ cases to be considered.

In h_1h_2 of these cases both events happen, in h_1f_2 cases the first happens and the second fails, in f_1h_2 cases the first fails and the second happens, and in f_1f_2 cases both fail. Let

$$p_1 = \frac{h_1}{h_1 + f_1}, \qquad \text{and} \qquad p_2 = \frac{h_2}{h_2 + f_2}.$$

(*a*) The chance that both events happen is

$$\frac{h_1h_2}{(h_1 + f_1)(h_2 + f_2)} = \frac{h_1}{h_1 + f_1} \times \frac{h_2}{h_2 + f_2} = p_1p_2.$$

(*b*) The chance that the first happens and second fails is

$$\frac{h_1f_2}{(h_1 + f_1)(h_2 + f_2)} = \frac{h_1}{h_1 + f_1} \times \frac{f_2}{h_2 + f_2} = p_1(1 - p_2).$$

(*c*) The chance that the first fails and second happens is

$$\frac{f_1h_2}{(h_1 + f_1)(h_2 + f_2)} = \frac{f_1}{h_1 + f_1} \times \frac{h_2}{h_2 + f_2} = (1 - p_1)p_2.$$

(*d*) The chance that both events fail is

$$\frac{f_1f_2}{(h_1 + f_1)(h_2 + f_2)} = \frac{f_1}{h_1 + f_1} \times \frac{f_2}{h_2 + f_2} = (1 - p_1)(1 - p_2).$$

Thus the probability that two independent events will both happen is equal to the product of their separate probabilities.

And, in general, if the probabilities of any number of independent events are $p_1, p_2, p_3, \cdots$, the probability that all the events will happen is $p_1p_2p_3 \cdots$.

Example 1. A bag contains four red, three white, and two blue balls. Three balls are drawn in succession, each being replaced before the next is drawn. What is the probability that the first ball drawn is red, the second white, and the third blue?

The probability that the first ball is red is 4/9; the probability that the second ball is white is 3/9, or 1/3; and the probability that the third ball is blue is 2/9.

Hence the probability of the happening of the three events in this order is

$$\frac{4}{9} \times \frac{1}{3} \times \frac{2}{9} = \frac{8}{243}.$$

EXAMPLE 2. What is the chance of throwing 7 in the first only of two successive throws with two dice?

The chance of throwing 7 at the first trial (Ex. 3, § 116) is 1/6.
The chance of not throwing 7 at the second trial is 5/6.
Hence the required probability is (1/6)(5/6) = 5/36.

119. **Dependent Events.** If the meaning of h_2 and f_2 in § 118 be slightly modified, the theorem with reference to the happening of two independent events may be used to determine the probability of the happening of two dependent events. Let h_2 and f_2 denote the number of ways in which the second event can happen, or fail, *after the first event has happened.*

Then the probability that both events will happen is

$$\frac{h_1 h_2}{(h_1 + f_1)(h_2 + f_2)} = p_1 p_2,$$

where p_2 denotes the probability that the second event will happen after the first has happened.

Thus the probability that two dependent events will both happen is equal to the product of the probability of the first, multiplied by the probability that when the first has happened the second will follow.

This theorem may be extended so as to apply to the probability of the happening of any number of dependent events with the modification mentioned above.

EXAMPLE. The first call letter of a broadcasting station is K. Three other letters are to be assigned to the station by drawing them in order from a box containing the twenty-six letters of the alphabet. If none of the letters are replaced, what is the probability that the station's "call" will be KUOA?

The probability that the first letter drawn is U is 1/26; the probability that the second letter is O is 1/25; and the probability that the third letter is A is 1/24.

Hence the probability that the "call" is KUOA is

$$\frac{1}{26} \times \frac{1}{25} \times \frac{1}{24} = \frac{1}{15{,}600}.$$

120. **Theorem Concerning Repeated Trials.** Let us now consider the probability of the happening of an event exactly r times in n trials.

Let p denote the probability of the happening of the event in one trial. Then $1 - p$ is the probability of its failing.

The probability that the event will happen in each of r trials and fail in each of the remaining $n - r$ trials is, by § 118, $p^r(1 - p)^{n-r}$.

Since the r trials can be selected out of the n trials in ${}_nC_r$ ways, we have derived the following theorem.

THEOREM. *If p is the probability that an event will happen in any single trial, then the probability that the event will happen exactly r times in n trials is ${}_nC_r p^r(1 - p)^{n-r}$.*

Thus the probability of throwing exactly 2 aces in 4 throws with a single die is

$${}_4C_2 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^2 = 6 \cdot \frac{1}{36} \cdot \frac{25}{36} = \frac{25}{216}.$$

EXERCISES

1. A bag contains six white balls and four black balls and six balls are successively drawn out and not replaced. What is the probability that they are alternately of different colors?

2. A, B, C, and D throw a die in succession until one throws an ace. What are their respective chances of throwing an ace at the first trial?

3. Find the probability of throwing doublets at least once in three throws with two dice.

4. If ten coins are tossed, what is the probability that at least four of them will be heads?

5. A box contains two white balls and four black ones. Four persons each draw a ball in succession and keep it. The first one who draws a white ball wins a prize. What are their respective chances of winning?

6. If four cards are drawn in succession from a pack, what is the probability that they will be of different suits?

7. In a series of games, the probability that the winner of any game wins the next game is 3/4. Find the probability that the winner of the first game wins three or more of the next four.

8. If p is the probability that an event will happen in one trial, show that the probability of its happening each time in n trials is p^n.

9. If p is the probability that an event will happen in one trial, show that the probability that it will happen at least r times in n trials is

$$p^n + {}_nC_1p^{n-1}(1-p) + {}_nC_2p^{n-2}(1-p)^2 + \cdots + {}_nC_{n-r}p^r(1-p)^{n-r}.$$

121. **Probability of Human Life.** We have seen that the probability that one or more events will happen can, in many cases, be determined by means of certain mathematical formulas. Let us now consider a problem which is not only of vital interest to every individual but also of great importance in fixing the amount of the annual premium to be paid on a life insurance policy or in answering the questions of any company or corporation which is working out a pension system for its employees; namely, the probability of human life. This problem can be solved, not by a mathematical formula, but by means of a so-called Experience Table of Mortality (p. 322) which is compiled from the averaged observations over a period of years of a large number of life insurance companies.

The following example will illustrate the use of this table.

EXAMPLE. A man of 30 marries a woman of 25. What is the probability that they will both be alive in 40 years?

The table shows that out of 85441 persons living at the age of 30, it may be expected that only 38569 will be alive at the age of 70. Hence the probability that the man will live to reach that age is 38569/85441, or 0.45, approximately.

The probability that the woman will live to reach the age of 65 is 49341/89032, or 0.55, approximately.

Hence the probability that both will be alive in 40 years is (§ 118) 0.45×0.55, or about 0.25

EXERCISES

1. What is the probability that a man 21 years of age will live to the age of 40? 50? 60? 70? 80?

2. A man is 38 and his son 16. What is the probability that both will live 40 years?

3. In Ex. 2, find the probability that at the end of 40 years (1) the son, but not the father, will be living; (2) the father, but not the son, will be living.

4. A man is 41 and his wife is 40. If they have been married for twenty years, what is the probability that they will live to celebrate their golden wedding?

5. A man is 46, his wife is 40, and his three sons are 16, 14, and 12, respectively. Find the probability that thirty years later (1) the entire family will be living; (2) the three sons, but neither parent, will be living.

6. A and B are twins just 12 years old. What is the probability that both will be living 50 years later?

CHAPTER XIV

DETERMINANTS

122. **Definitions.** We have seen (§ 11) that the solution of a system of linear equations may be very conveniently expressed by means of certain symbols, called ***determinants,*** defined as follows:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1,$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$$

The numbers a_1, a_2, a_3, b_1, $\cdots$, are called the *elements* of the determinant. Those elements which lie in a horizontal line form a *row*, and those which lie in a vertical line form a *column*, of the determinant. Those elements which lie along the diagonal through the upper left hand corner form the *principal diagonal;* those which lie along the other diagonal form the *minor diagonal.*

In the above determinants the elements of the first row are distinguished by the subscript 1, those of the second row by the subscript 2, etc. The elements of the first column are distinguished by the letter a, those of the second column by the letter b, etc.

The first determinant above contains 2^2, or 4 elements and is said to be of the *second order;* the second one contains 3^2, or 9 elements and is of the *third order.* In general, a determinant of n rows and n columns contains n^2 elements and is of the *nth order.*

We have explained (§ 11) a very simple method of finding the expanded form of a determinant of the second or third order. It should be observed, however, that this method does not apply to determinants of higher than the third order.

EXERCISES

Find the value of each of the following determinants.

1. $\begin{vmatrix} 3 & 5 \\ 2 & 6 \end{vmatrix}$. 2. $\begin{vmatrix} 4 & -1 \\ 3 & 2 \end{vmatrix}$. 3. $\begin{vmatrix} 2 & 5 \\ 1 & -1 \end{vmatrix}$. 4. $\begin{vmatrix} -1 & 5 \\ -2 & -12 \end{vmatrix}$.

5. $\begin{vmatrix} 1 & 2 & 4 \\ -2 & -1 & -4 \\ 3 & -3 & 5 \end{vmatrix}$. 6. $\begin{vmatrix} -4 & 8 & 15 \\ 6 & -5 & -4 \\ 14 & 11 & 9 \end{vmatrix}$. 7. $\begin{vmatrix} 3 & 2 & 3 \\ 5 & -1 & 5 \\ 1 & 4 & 1 \end{vmatrix}$.

Verify the following by expanding the determinants.

8. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

9. $\begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

10. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$.

11. $\begin{vmatrix} a_1 + x_1 & b_1 & c_1 \\ a_2 + x_2 & b_2 & c_2 \\ a_3 + x_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{vmatrix}$.

123. **Inversions.** If, in any permutation of the numbers 1, 2, 3, 4, $\cdots$, (or the letters a, b, c, d, $\cdots$,) a number (or letter) precedes one which it should naturally follow, there is said to be an ***inversion of order.***

Thus, in the permutation 2, 4, 1, 3, there are three inversions: 2 before 1, 4 before 1, and 4 before 3. Also, in the permutation a, c, d, b, there are two inversions: c before b, and d before b.

124. **The Expanded Form of Any Determinant.** Let us consider the expression

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2,$$

which is the expanded form of the typical determinant of the third order shown in § 122. The following facts are obvious:

(*a*) *Each of the terms is the product of three elements.*

(*b*) *Each product contains one and only one element from each row;* that is, one with a subscript 1, one with a subscript 2, and one with a subscript 3.

(*c*) *Each product contains one and only one element from each column;* that is, one a, one b, and one c.

(*d*) *If a term is preceded by the sign +, the number of inversions in the subscripts is* ***even;*** *and if a term is preceded by the sign −, the number of inversions in the subscripts is* ***odd.***

Thus, there are 2 inversions in the subscripts in the term $+ a_3b_1c_2$, and 3 inversions in the term $- a_3b_2c_1$.

We can, therefore, obtain all the terms of the above expression by writing the letters a, b, c in their natural order and attaching the subscripts 1, 2, 3 in every possible order, determining the sign of each term by counting the number of inversions in the subscripts. The number of terms is evidently ${}_3P_3$, or 6.

Similarly, it will be convenient to define the typical determinant of the fourth order,

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

as the algebraic sum of all the products of the form $a_1b_2c_3d_4$ which may be obtained by keeping the letters in their natural order and writing the subscripts in all possible orders, the

sign before any product being + if the number of inversions is even, and − if the number is odd.

Since there are 4!, or 24 permutations of four things taken all at a time, there will be 24 terms in the expanded form of a determinant of the fourth order.

We shall now define the general determinant of the nth order,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 & \cdots & m_1 \\ a_2 & b_2 & c_2 & \cdots & m_2 \\ a_3 & b_3 & c_3 & \cdots & m_3 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_n & b_n & c_n & \cdots & m_n \end{vmatrix},$$

as the algebraic sum of all the products of the form $a_1b_2c_3 \cdots m_n$ which may be obtained by keeping the letters in their natural order and writing the subscripts in all possible orders, the sign before any product being + if the number of inversions is even, and − if the number is odd.

There are $n!$ terms in the expanded form of a determinant of the nth order, and each term contains one and only one element from each row and from each column.

With this definition in mind we may now express a determinant in a very convenient form by writing only the elements of the principal diagonal. Thus the typical determinant of the fourth order may be written $|a_1b_2c_3d_4|$.

125. **Rearrangement of Elements in a Term.** If two adjacent elements in any term of the expanded form of the determinant $|a_1b_2c_3 \cdots m_n|$ be interchanged, the number of inversions in the subscripts in that term is obviously changed by 1. This interchange also introduces one inversion in the letters. Hence the combined number of inversions among the subscripts and among the letters is changed by either 2 or 0.

Thus, if we write the term $a_1b_2c_3 \cdots m_n$ in the form $a_1c_3b_2 \cdots m_n$, we introduce one more inversion among the subscripts (3 before

2), and one inversion among the letters (c before b), without making any other change in the number of inversions among the letters or subscripts of the original arrangement. That is, we have added 2 to the number of inversions among the letters and subscripts.

Since any rearrangement can be made from the original arrangement by a series of interchanges of adjacent elements, we have derived the following theorem.

THEOREM. *If the elements in a term of a determinant are rearranged in any order, the combined number of inversions of letters and subscripts is changed by an even number.*

Let us now rearrange the elements in each term of the expansion,

$$(1) \quad a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2,$$

so that the subscripts are written in their natural order. We obtain

$$(2) \quad a_1b_2c_3 + c_1a_2b_3 + b_1c_2a_3 - c_1b_2a_3 - b_1a_2c_3 - a_1c_2b_3.$$

The form (1) contains no inversion of letters and the form (2) contains no inversion of subscripts. Then, by the above theorem, if the number of inversions among the subscripts of any term of (1) is even (or odd), the number of inversions among the letters of the corresponding term of (2) is even (or odd).

Since a rearrangement of the elements of a term has no effect on the sign of that term, it is evident that the expanded form of the determinant may be obtained by keeping the subscripts in their natural order and arranging the letters in all possible orders, the sign of each term being $+$, or $-$, according as the number of inversions among the letters is even, or odd.

It should be observed that the preceding statement, which applies to a determinant of any order, merely states that an

interchange of roles of letters and subscripts makes no difference in the definition of a determinant.

EXERCISES

1. How many terms are there in the expanded form of a determinant of the fifth order? sixth order?

2. How many inversions of subscripts are there in the arrangement $c_5b_2a_1d_6e_3f_4$? How many inversions of letters?

3. Write the expanded form of the determinant $|a_1b_2c_3d_4|$.

4. Try to expand the determinant $|a_1b_2c_3d_4|$ by the special method of § 11. How many terms do you get?

By assigning proper values to the elements in the determinant of Ex. 3 find the value of

5. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 6 & 11 & 25 & 7 \\ 8 & 7 & 1 & 3 \\ 1 & 6 & 6 & 5 \end{vmatrix}$. **6.** $\begin{vmatrix} 1 & 2 & 2 & 4 \\ 1 & 4 & 4 & 1 \\ 1 & 1 & 2 & 2 \\ 4 & 8 & 11 & 13 \end{vmatrix}$.

7. In a determinant of the fifth order what sign should be given to the following terms?

$$a_2b_4c_1d_3c_5, \quad a_5b_2c_1d_3c_4, \quad a_2b_4c_5d_3e_1,$$
$$a_2b_1c_3d_5e_4, \quad a_5b_4c_2d_1e_3, \quad a_1b_4c_2d_3e_5.$$

126. **Properties of Determinants.** We shall now state and prove certain important properties of determinants. For convenience we shall in most cases limit our discussions to determinants of the third order, although similar proofs may be given for determinants of any order.

(1) *The value of a determinant is unchanged if its rows are changed to columns and its columns to rows.*

Consider the determinants

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

We may obtain D' from D by merely interchanging the roles of the letters and subscripts. Hence, by § 125, $D = D'$.

As a corollary of the above theorem we may say that *any theorem which is true for rows of a determinant is also true for columns, and vice-versa.*

(2) *If any two columns (or rows) of a determinant are interchanged, the sign of the determinant is changed.*

Let us first interchange two adjacent rows, say the second and third rows. This merely amounts to interchanging the subscripts 2 and 3 in each term of the expansion of the determinant. This will change the number of inversions among the subscripts by 1 and, consequently, will change the sign of every term in the expansion. Hence the theorem is true if the rows (or columns) are adjacent.

For example, the term $a_2b_1c_3$ becomes $a_3b_1c_2$ and one additional inversion (3 before 2) is added without disturbing the other inversions. Hence these terms have opposite signs.

Let us now suppose that there are k rows between the two rows R and R' which are to be interchanged. We can make the row R' adjacent to the row R by passing R' successively over the k intervening rows, thus changing the sign of the determinant k times. If we now pass the row R successively over R' and the k originally intervening rows, which changes the sign of the determinant $k + 1$ times, we have effected the interchange of the non-adjacent rows R and R' by changing the sign of the determinant $k + k + 1$, or $2k + 1$ times. Since $2k + 1$ is always an odd number, the interchange of R and R' changes the sign of the determinant.

(3) *If two columns (or rows) of a determinant are identical, its value is zero.*

Let D be any determinant with two rows identical. If these rows be interchanged, the form of the determinant is not altered. but, by (2), the interchange of two rows changes the sign of the determinant. Hence $D = -D$, or $2D = 0$. Then $D = 0$.

(4) *If each element in any column (or row) of a determinant be multiplied by the same number, the determinant is multiplied by this number.*

This follows at once from the fact that each term of the expanded form of a determinant contains one and only one element from a given column (or row).

If any number is a common factor of all the elements of any column (or row), it is a factor of the determinant.

Thus, if the factor 4 be removed from the first row and the factor 3 from the second column, we have

$$\begin{vmatrix} 4 & 12 & 8 \\ -2 & 9 & 4 \\ 1 & -6 & 1 \end{vmatrix} = 4 \begin{vmatrix} 1 & 3 & 2 \\ -2 & 9 & 4 \\ 1 & -6 & 1 \end{vmatrix} = 12 \begin{vmatrix} 1 & 1 & 2 \\ -2 & 3 & 4 \\ 1 & -2 & 1 \end{vmatrix}.$$

(5) *If each element of a column (or row) is the sum of two numbers, the determinant may be expressed as the sum of two determinants.* That is,

$$\begin{vmatrix} a_1 + x_1 & b_1 & c_1 \\ a_2 + x_2 & b_2 & c_2 \\ a_3 + x_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{vmatrix}.$$

The term $(a_1 + x_1)b_2c_3$ of the first determinant may evidently be written $a_1b_2c_3 + x_1b_2c_3$, which is the sum of the corresponding terms of the last two. Similarly, every term of the first determinant is the sum of the corresponding terms of the last two.

(6) *If all the elements in any column (or row) be multiplied by the same number, and either added to, or subtracted from, the corresponding elements in another column (or row), the value of the determinant is not changed.* That is,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + mb_1 & b_1 & c_1 \\ a_2 + mb_2 & b_2 & c_2 \\ a_3 + mb_3 & b_3 & c_3 \end{vmatrix}.$$

Let D denote the first determinant and D' the second. Then, by (5), D' is the sum of two determinants, of which the first is D and the second (after taking out the factor m) is zero, by (3). Hence $D' = D$.

127. **Simplification of Determinants.** By making use of the theorems of § 126 it is frequently possible to reduce a given determinant to a simpler form. The following examples will illustrate some of the methods used.

Example 1. Find the value of the determinant

$$D = \begin{vmatrix} 4 & 14 & -4 \\ 4 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix}.$$

$$D = 2\begin{vmatrix} 2 & 7 & -2 \\ 4 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 & -2 \\ 4 & -11 & 1 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 1 & 0 \\ 4 & -11 & 5 \\ 1 & 0 & 0 \end{vmatrix} = 2 \times 5 = 10.$$

Explanation. We first took the factor 2 out of the first row. We then multiplied all the elements in the first column of the resulting determinant by 3 and subtracted them from the corresponding elements in the second column. We then added the elements in the first column to the corresponding elements in the third column. This reduces the given determinant to a determinant in which two elements in one row are zero. Its value may now be quickly found by the method of § 11.

Example 2. Factor the determinant

$$D = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}.$$

If $a = b$, the first and second columns are identical; then $D = 0$, and $a - b$ is a factor of D. Similarly when $a = c$, and when $b = c$, the determinant has two columns identical, and $D = 0$. Thus $a - c$ and $b - c$ are factors of D. Since the expanded form of the determinant is of the third degree in the letters a, b, and c, there are no other factors except perhaps a numerical factor. Then $D = k(a - b)(a - c)(b - c)$.

Since the coefficient of the term a^2b in the expansion of the determinant is -1, it is obvious that $k = -1$. Then

$$D = -(a - b)(a - c)(b - c).$$

EXERCISES

Express each of the following as a determinant in which two of the elements in a column or row are zero.

1. $\begin{vmatrix} 1 & 3 & -1 \\ 2 & 4 & 1 \\ -2 & 1 & 3 \end{vmatrix}$. **2.** $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 5 & 2 \\ -1 & 6 & 3 \end{vmatrix}$. **3.** $\begin{vmatrix} a & b & a+b \\ b & c & b+c \\ c & a & c+a \end{vmatrix}$.

4. Why is it not possible, in general, to express a given determinant as a determinant in which all the elements in a column or row are zero?

Find the value of the following determinants, making use of one or more of the theorems of § 126.

5. $\begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & -1 \\ 1 & 1 & 2 \end{vmatrix}$. **6.** $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 5 & 8 \\ 3 & 7 & 11 \end{vmatrix}$. **7.** $\begin{vmatrix} 1 & -1 & 3 \\ 4 & -2 & 1 \\ 3 & 2 & -1 \end{vmatrix}$.

8. Without expanding the determinant, show that the equation here given is of the second degree in x, and that one of its roots is $x = 1$. What is the other root?

$$\begin{vmatrix} x^2 & 2x & 1 \\ 1 & 2 & 1 \\ 9 & 6 & 1 \end{vmatrix} = 0.$$

9. Without expanding the determinant, show that the equation here given is of the first degree in x and y, and that it is satisfied by the pair of values $x = 4$, $y = 1$. What other pair of values will satisfy the equation?

$$\begin{vmatrix} x & y & 1 \\ 4 & 1 & 1 \\ 2 & -3 & 1 \end{vmatrix} = 0.$$

10. Show that

$$\begin{vmatrix} x & y & 1 \\ 3 & 4 & 1 \\ -2 & 1 & 1 \end{vmatrix} = 0$$

is the equation of the straight line through the points (3, 4) and (− 2, 1).

11. Write in determinant form the equation of the straight line which passes through the points (2, 4) and (1, − 5); (− 3, 0) and (0, 2); (0, 0) and (4, 5); (x_1, y_1) and (x_2, y_2).

Factor each of the following determinants.

12. $\begin{vmatrix} 1 & a & a^2 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{vmatrix}$. **13.** $\begin{vmatrix} a & b & 1 \\ a^2 & b^2 & 1 \\ 3 & 3 & 3 \end{vmatrix}$. **14.** $\begin{vmatrix} x & x^2 & 2 \\ y & y^2 & 2 \\ z & z^2 & 2 \end{vmatrix}$.

15. $\begin{vmatrix} 1 & a & b \\ 1 & a^2 & b^2 \\ 1 & a^3 & b^3 \end{vmatrix}$. **16.** $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ac \\ c & c^2 & ab \end{vmatrix}$. **17.** $\begin{vmatrix} a & b+c & a^2 \\ b & a+c & b^2 \\ c & a+b & c^2 \end{vmatrix}$.

18. Show that $a + b + c$ is a factor of the determinant

$$\begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix}.$$

Solve the following equations:

19. $\begin{vmatrix} 2 & 2 & x \\ 3 & 3 & 3 \\ 4 & x & 4 \end{vmatrix} = 0.$ **20.** $\begin{vmatrix} 3-x & 7+2x & 11-3x \\ 5+x & 3-2x & 17+3x \\ 1 & 2 & 3 \end{vmatrix} = 0.$

Prove by means of § 126 that the value of each of the following determinants is 0.

21. $\begin{vmatrix} c & a & d & b \\ a & c & d & b \\ a & c & b & d \\ c & a & b & d \end{vmatrix}.$ **22.** $\begin{vmatrix} 1 & a & b & c+d \\ 1 & b & c & a+d \\ 1 & c & d & a+b \\ 1 & d & a & b+c \end{vmatrix}.$

128. **Minors.** If the column and row which intersect at a certain element, say b_k, be erased, the determinant remaining, which is of order one lower than the given determinant, is called a ***minor*** of b_k and is denoted by B_k.

Thus in the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

the minors of the elements b_2, a_3, and c_1 are

$$B_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}, \qquad A_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \qquad C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

129. **Expansion by Minors.** The expanded form of the typical determinant of the third order,

(1) $\quad a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2,$

may be written in the form

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1),$$

or

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix},$$

which, by § 128, may be written in the form

$$a_1A_1 - a_2A_2 + a_3A_3.$$

Then

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1 - a_2A_2 + a_3A_3.$$

When written in this way the determinant is said to be *expanded by the minors of its first column.*

Again, the expansion (1) may be written in the form

$$-a_2(b_1c_3 - b_3c_1) + b_2(a_1c_3 - a_3c_1) - c_2(a_1b_3 - a_3b_1),$$

or

$$-a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}.$$

Hence if the third order determinant be expanded by the minors of its second row, we have

$$D = -a_2A_2 + b_2B_2 - c_2C_2.$$

In a similar way, it may be shown that a determinant of the third order may be expanded by the minors of any given column or row, the sign of each of the various products of elements by their minors being determined by the following rule.

The product of the element in the mth column and the nth row by its minor is to be taken as positive or negative according as $m + n$ is even or odd.

Thus

$$\begin{aligned} D &= a_3A_3 - b_3B_3 + c_3C_3 \\ &= c_1C_1 - c_2C_2 + c_3C_3 \\ &= -b_1B_1 + b_2B_2 - b_3B_3. \end{aligned}$$

By rearranging the terms in the expansion of a determinant of the fourth, or higher, order it may be shown that a determinant of any order may be expanded by the minors of any column or row, the sign of the product of any element by its minor being determined by the foregoing rule. The details of the proof will be omitted for the sake of brevity.

Thus, if D is the typical determinant of the fourth order,

$$\begin{aligned} D &= a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4 \\ &= c_1C_1 - c_2C_2 + c_3C_3 - c_4C_4 \\ &= -a_2A_2 + b_2B_2 - c_2C_2 + d_2D_2. \end{aligned}$$

The expansion of a determinant by the minors of a column or row furnishes a very convenient method for finding the value of any given determinant.

EXAMPLE. Find the value of the determinant

$$\begin{vmatrix} 1 & 3 & 1 & 0 \\ 3 & 2 & 3 & 1 \\ 0 & 2 & 5 & 1 \\ 2 & 2 & 2 & 3 \end{vmatrix}.$$

First subtract twice the fourth column from the second and five times the fourth column from the third. We then obtain the following equivalent determinant, having three zeros in its third row.

$$\begin{vmatrix} 1 & 3 & 1 & 0 \\ 3 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & -4 & -13 & 3 \end{vmatrix}.$$

Now expand this determinant by the minors of its third row:

$$0 \times \begin{vmatrix} 3 & 1 & 0 \\ 0 & -2 & 1 \\ -4 & -13 & 3 \end{vmatrix} - 0 \times \begin{vmatrix} 1 & 1 & 0 \\ 3 & -2 & 1 \\ 2 & -13 & 3 \end{vmatrix}$$

$$+ 0 \times \begin{vmatrix} 1 & 3 & 0 \\ 3 & 0 & 1 \\ 2 & -4 & 3 \end{vmatrix} - 1 \times \begin{vmatrix} 1 & 3 & 1 \\ 3 & 0 & -2 \\ 2 & -4 & -13 \end{vmatrix}.$$

The first three terms will vanish because of the factor 0 in each. Thus the original fourth order determinant reduces to a single determinant of the third order. Let us now find the value of this determinant.

Subtracting three times the first row from the second row and twice the first row from the third row, we obtain

$$-1 \times \begin{vmatrix} 1 & 3 & 1 \\ 3 & 0 & -2 \\ 2 & -4 & -13 \end{vmatrix} = -1 \times \begin{vmatrix} 1 & 3 & 1 \\ 0 & -9 & -5 \\ 0 & -10 & -15 \end{vmatrix}$$

$$= (-1)(-5) \times \begin{vmatrix} 1 & 3 & 1 \\ 0 & -9 & -5 \\ 0 & 2 & 3 \end{vmatrix}.$$

Now expand this determinant by the minors of its first column.

$$\begin{vmatrix} 1 & 3 & 1 \\ 0 & -9 & -5 \\ 0 & 2 & 3 \end{vmatrix} = 1 \times \begin{vmatrix} -9 & -5 \\ 2 & 3 \end{vmatrix} - 0 \times \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix}$$

$$+ 0 \times \begin{vmatrix} 3 & 1 \\ -9 & -5 \end{vmatrix} = \begin{vmatrix} -9 & -5 \\ 2 & 3 \end{vmatrix}.$$

Thus the value of the original determinant is

$$5 \times \begin{vmatrix} -9 & -5 \\ 2 & 3 \end{vmatrix} = 5(-27 + 10) = -85.$$

130. **Simultaneous Equations.** In § 11 we found that systems of simultaneous linear equations between two (or three) unknown quantities could be readily solved by making use of determinants of the second (or third) order. The method of § 11 is a general one and applies to any set of simultaneous linear equations containing as many equations as unknown quantities. For brevity, the proof will not be given.

EXERCISES

Find the value of each of the following determinants by expanding by minors.

1. $\begin{vmatrix} 30 & 11 & 20 & 38 \\ 6 & 3 & 0 & 9 \\ 11 & -2 & 36 & 3 \\ 19 & 6 & 17 & 22 \end{vmatrix}$.

2. $\begin{vmatrix} 5 & 7 & 8 & 6 \\ 11 & 16 & 13 & 11 \\ 14 & 24 & 20 & 23 \\ 7 & 13 & 12 & 2 \end{vmatrix}$.

3. $\begin{vmatrix} 6 & 15 & 11 & 10 \\ 5 & 16 & 12 & 9 \\ 7 & 14 & 10 & 11 \\ 3 & 18 & 9 & 12 \end{vmatrix}$.

4. $\begin{vmatrix} 2 & 7 & -2 & 8 \\ 4 & 1 & 1 & -3 \\ 0 & 3 & -1 & 4 \\ 6 & 4 & 2 & -8 \end{vmatrix}$.

5. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & a & a \\ a & a & b & a \\ a & a & a & b \end{vmatrix}$.

6. $\begin{vmatrix} m & y & n & x \\ x & y & n & m \\ x & n & y & m \\ m & n & y & x \end{vmatrix}$.

Solve the following systems of equations making use of the method of § 11.

7.
$$\begin{cases} a + 3b + 4c = 14, \\ a + 2b + c = 7, \\ 2a + b + 2c = 2. \end{cases}$$

8.
$$\begin{cases} ax + by + cz = a, \\ bx + cy + az = b, \\ cx + ay + bz = c. \end{cases}$$

9.
$$\begin{cases} 3x + y + z = 20, \\ 3v + x + 4y = 30, \\ 3v + 6x + z = 40, \\ 5v + 8y + 3z = 50. \end{cases}$$

10.
$$\begin{cases} x + y + z + u = 1, \\ 2x + 3y - 4z + 5u = -31, \\ 3x - 4y + 5z + 6u = -22, \\ 4x + 5y - 6z - u = -13. \end{cases}$$

CHAPTER XV

PARTIAL FRACTIONS

131. **Definitions.** We have seen (§ 7) that it is always possible to add two or more algebraic fractions and obtain a single fraction whose denominator is the lowest common multiple of the denominators of the several fractions.

Thus

$$\frac{2}{x} + \frac{x - 2}{x^2 - 2x + 2} = \frac{3x^2 - 6x + 4}{x^3 - 2x^2 + 2x}.$$

In advanced mathematics, especially in the study of the calculus, the inverse problem is of great importance, namely:

Given an algebraic fraction in its lowest terms, required to find two or more fractions whose sum is the given fraction.

The fraction is then said to be separated into *partial fractions.*

Thus

$$\frac{8x - 6}{4x^2 - 9} = \frac{3}{2x + 3} + \frac{1}{2x - 3}.$$

A *proper fraction* is one in which the degree of the numerator is less than that of the denominator. Any fraction can be reduced, by the process of division, to the sum of a polynomial and a proper fraction.

If the denominator of a fraction can be resolved into factors, the fraction can always be written as the sum of several fractions whose denominators are factors of the given denominator. We shall consider only the case where the factors are real and of the first or second degree.

It should be observed that if an algebraic fraction be

separated into partial fractions, the resulting equation is an *identity* (§ 9); that is, it is true for all values of the unknowns for which the expressions can be evaluated.

132. **Theorem Concerning Identities.** We shall now prove an important theorem with reference to identities.

THEOREM. *If two polynomials in x, one of degree n and the other of degree n or less, are equal for more than n values of x, the coefficients of like powers of x in the two polynomials are equal.*

Suppose

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = b_0x^n + b_1x^{n-1} + \cdots + b_n$$

for more than n values of x, then the equation

$$(a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + \cdots + (a_n - b_n) = 0,$$

which is of degree n, has more than n roots. But this is possible when and only when each of its coefficients is equal to zero; that is, when $a_0 = b_0$, $a_1 = b_1$, $\cdots$, $a_n = b_n$.

In this case the equation reduces to the form

$$0x^n + 0x^{n-1} + \cdots + 0 = 0,$$

which is true for all values of x.

Thus, if $4x^2 + x - 3 = Ax^2 + (A - B)x + B + C$ for more than two values of x, then $A = 4$, $A - B = 1$, and $B + C = -3$.

By making use of the above theorem a fraction may be easily separated into partial fractions. In every case we assume that the fraction can be expressed in a certain form and then find the values of certain constants which are left undetermined.

133. **Factors of the Denominator Not Repeated.** The following examples will illustrate several methods which may be used in this case.

EXAMPLE 1. Separate into partial fractions

$$\frac{5x^2 - 5x + 2}{(x - 1)(x - 2)(x + 1)}.$$

Since the factors in the denominator are linear, the numerators of each of the fractions corresponding to these factors must be constants, so that each partial fraction will be a proper fraction. Let us assume that

$$\frac{5x^2 - 5x + 2}{(x - 1)(x - 2)(x + 1)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x + 1},$$

where A, B, C are constants whose values are to be determined.

Clearing of fractions gives

$$5x^2 - 5x + 2 = A(x - 2)(x + 1) + B(x - 1)(x + 1) + C(x - 1)(x - 2).$$

This equation is an identity, that is, it is true for all values of x. Hence it must be true when $x = 1$, 2, or -1.

When $x = 1$, the second and third terms vanish and the identity reduces to $5 - 5 + 2 = -2A$. Then $A = -1$.

When $x = 2$, the first and third terms vanish and we obtain $20 - 10 + 2 = 3B$. Then $B = 4$.

When $x = -1$, the first and second terms vanish and we obtain $5 + 5 + 2 = 6C$. Then $C = 2$.

Hence

$$\frac{5x^2 - 5x + 2}{(x - 1)(x - 2)(x + 1)} = \frac{-1}{x - 1} + \frac{4}{x - 2} + \frac{2}{x + 1}.$$

CHECK. The student should now check this result by adding the fractions in the right member.

EXAMPLE 2. Separate into partial fractions

$$\frac{5x^2 - 75}{x^3 - 25x}.$$

Assume

$$\frac{5x^2 - 75}{x^3 - 25x} = \frac{A}{x} + \frac{B}{x - 5} + \frac{C}{x + 5}.$$

Clearing of fractions and collecting terms gives

$$\begin{aligned} 5x^2 - 75 &= A(x-5)(x+5) + Bx(x+5) + Cx(x-5) \\ &= (A+B+C)x^2 + (5B-5C)x - 25A. \end{aligned}$$

Since this equation is true for all values of x, the coefficients of like powers of x must be equal (§ 132). Then

$$A + B + C = 5, \qquad 5B - 5C = 0, \qquad 25A = 75.$$

Solving this system of equations gives $A = 3$, $B = 1$, $C = 1$. Hence

$$\frac{5x^2 - 75}{x^3 - 25x} = \frac{3}{x} + \frac{1}{x+5} + \frac{1}{x-5}.$$

EXAMPLE 3. Separate into partial fractions

$$\frac{x^2 - 6x - 1}{(x-1)(x^2+x+1)}.$$

The numerator of the partial fraction which corresponds to the factor $x^2 + x + 1$ may contain the first power of x, but no higher power. Let us assume that

$$\frac{x^2 - 6x - 1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}.$$

Then

$$\begin{aligned} x^2 - 6x - 1 &= A(x^2+x+1) + (Bx+C)(x-1) \\ &= (A+B)x^2 + (A-B+C)x + A - C. \end{aligned}$$

Equating coefficients of like powers of x, we find

$$A + B = 1, \qquad A - B + C = -6, \qquad A - C = -1.$$

Solving these equations, we have $A = -2$, $B = 3$, $C = -1$. Hence

$$\frac{x^2 - 6x - 1}{(x-1)(x^2+x+1)} = -\frac{2}{x-1} + \frac{3x-1}{x^2+x+1}.$$

EXERCISES

Separate the following into partial fractions.

1. $\dfrac{x+7}{x^2-2x-15}$.
2. $\dfrac{43x+6}{6x^2+5x-6}$.
3. $\dfrac{7x-1}{6x^2-5x+1}$.
4. $\dfrac{1+3x+2x^2}{(1-2x)(1-x^2)}$.
5. $\dfrac{62x-38}{(4x-3)(6x^2+x-2)}$.
6. $\dfrac{2x+3}{x^3+x^2-2x}$.
7. $\dfrac{3x^2-2}{(x+1)(x^2+x+1)}$.
8. $\dfrac{22x-6}{8x^3-27}$.
9. $\dfrac{6x+26}{(x-4)(3x^2+2)}$.

134. **Factors of the Denominator Repeated.** If one or more of the factors of the denominator are repeated, the assumed form is taken as in the following examples. The method used in the first example is very convenient when the denominator is in the form $(x-a)^n$.

EXAMPLE 1. Separate into partial fractions

$$\frac{x^2-5x+7}{(x-2)^3}.$$

Substitute $y+2$ for x. Then

$$\frac{x^2-5x+7}{(x-2)^3}=\frac{(y+2)^2-5(y+2)+7}{y^3}=\frac{y^2-y+1}{y^3}$$

$$=\frac{1}{y}-\frac{1}{y^2}+\frac{1}{y^3}=\frac{1}{x-2}-\frac{1}{(x-2)^2}+\frac{1}{(x-2)^3}.$$

EXAMPLE 2. Separate into partial fractions

$$\frac{5x^2+4x-5}{(x-1)^2(x-3)}.$$

Assume

$$\frac{5x^2+4x-5}{(x-1)^2(x-3)}=\frac{A}{(x-1)^2}+\frac{B}{x-1}+\frac{C}{x-3}.$$

Then

$$5x^2+4x-5=A(x-3)+B(x-1)(x-3)+C(x-1)^2$$
$$=(B+C)x^2+(A-4B-2C)x-3A+3B+C.$$

Equating coefficients of like powers of x gives

$$B + C = 5, \quad A - 4B - 2C = 4, \quad -3A + 3B + C = -5.$$

Solving these equations gives $A = -2$, $B = -8$, $C = 13$. Hence

$$\frac{5x^2 + 4x - 5}{(x-1)^2(x-3)} = \frac{-2}{(x-1)^2} - \frac{8}{x-1} + \frac{13}{x-3}.$$

135. **Improper Fractions.** If the numerator of the given fraction is of higher degree than the denominator, we may either divide the numerator by the denominator until a remainder is obtained which is of lower degree than the denominator, or use the method of Example 2 below.

EXAMPLE 1. Separate into an integral expression and partial fractions

$$\frac{x^4 + 2x^2 + x - 1}{x^3 + 2x^2 - x - 2}.$$

We find by division

$$\frac{x^4 + 2x^2 + x - 1}{x^3 + 2x^2 - x - 2} = x - 2 + \frac{7x^2 + x - 5}{x^3 + 2x^2 - x - 2}.$$

Also

$$\frac{7x^2 + x - 5}{x^3 + 2x^2 - x - 2} = \frac{7}{x+2} + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}.$$

Hence

$$\frac{x^4 + 2x^2 + x - 1}{x^3 + 2x^2 - x - 2} = x - 2 + \frac{7}{x+2} + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}.$$

EXAMPLE 2. Separate into an integral expression and partial fractions

$$\frac{x^4 + x^3 - 8x^2 - 4}{x^2 - 3x + 2}.$$

Since the first term of the quotient is x^2, it will be convenient to assume that

$$\frac{x^4 + x^3 - 8x^2 - 4}{x^2 - 3x + 2} = x^2 + Ax + B + \frac{C}{x-1} + \frac{D}{x-2}.$$

Clearing of fractions, we have

$$x^4 + x^3 - 8x^2 - 4 = x^4 - 3x^3 + 2x^2 + A(x^3 - 3x^2 + 2x) + B(x^2 - 3x + 2) + C(x - 2) + D(x - 1),$$

or

$$4x^3 - 10x^2 - 4 = Ax^3 + (B - 3A)x^2 + (2A - 3B + C + D)x + 2B - 2C - D.$$

Then

$$A = 4, \qquad 2A - 3B + C + D = 0,$$
$$B - 3A = -10, \qquad 2B - 2C - D = -4.$$

Solving these equations gives $A = 4$, $B = 2$, $C = 10$, $D = -12$. Hence

$$\frac{x^4 + x^3 - 8x^2 - 4}{x^2 - 3x + 2} = x^2 + 4x + 2 + \frac{10}{x - 1} - \frac{12}{x - 2}.$$

EXERCISES

Separate the following into partial fractions.

1. $\dfrac{x - 4}{(x + 1)^3}$.
2. $\dfrac{x^2 - x + 2}{(x - 1)^4}$.
3. $\dfrac{8x^2}{(2x - 3)^3}$.
4. $\dfrac{9}{(x - 1)(x - 2)^2}$.
5. $\dfrac{2x^2 - 2x + 1}{x^2(x - 1)^2}$.
6. $\dfrac{1}{x^2(x^2 + 1)}$.

Separate the following into integral expressions and partial fractions.

7. $\dfrac{6x^3 + 5x^2 - 7}{(3x + 1)(x - 1)}$.
8. $\dfrac{2x^6 - 2x^5 - 3x^2 + 1}{x^4 - x^3}$.

CHAPTER XVI

INTEREST AND ANNUITIES

136. **Interest.** Interest is money paid for the use of money. It may be regarded as *rent* which is paid for the use of borrowed capital for a period of time. The borrowed capital is called the *principal*, the rate percent which is paid for the use of the principal is called the *rate of interest*. The *amount* is the sum of the principal and the interest.

137. **Simple Interest.** Simple interest is interest which is computed on the original principal for the entire time for which the money was borrowed. Simple interest is paid at the end of a specified *interest period* which is usually one year. The following formulas are useful. Let P = principal, r = rate of interest, n = number of years, A = amount, and I = interest. Then

$$I = Pnr,$$

and

$$A = P + I = P + Pnr$$

or

$$A = P(1 + nr).$$

If equal principals P are invested at the same rate r, for two different periods of time n_1 and n_2, then the interests I_1 and I_2 are

$$I_1 = Pn_1r, \qquad \text{and} \qquad I_2 = Pn_2r,$$

and

$$\frac{I_1}{I_2} = \frac{n_1}{n_2}.$$

Thus for simple interest, the interest is proportional to the time of investment.

138. **Simple Amount—Time Graph.** If we consider a fixed principal P and a fixed rate r, the equation

$$A = P(1 + nr)$$

is a linear equation in A and n, and its graph is a straight line.

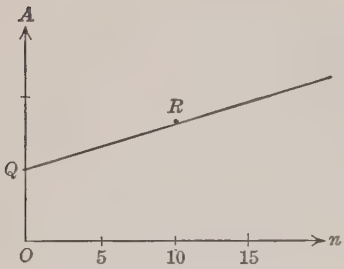


FIG. 34

Thus if $P = \$1$, and $r = 6\%$,

$$A = .06n + 1.$$

The graph of this equation is shown in the figure. We can draw the graph through the two points $Q(0, 1)$ and $R(10, 1.6)$.

139. **Present Value. Simple Discount.** The principal that would amount to a fixed sum, say A dollars, in n interest periods, at a fixed rate r, is called the *present value* of the amount A. Since

$$A = P(1 + nr),$$

we have

$$P = \frac{A}{1 + nr},$$

which is the formula for the present value.

The *simple discount* is the difference between the amount A and the present value P. Thus

$$D = A - P = A - \frac{A}{1 + nr} = \frac{Anr}{1 + nr}.$$

Thus the present value of \$100 due in one year if the rate of interest is 6% is

$$P = \frac{100}{1.06} = \$94.34,$$

and the discount is $D = \$100 - \$94.34 = \$5.66$

The *rate of discount* is the discount for the unit of time on the unit principal.

Thus in the exercise above the rate of discount is

$$\frac{\$5.66}{\$100} = .0566 = 5.66\%.$$

If D is the discount, A the amount, n the time, and d the rate of discount, then

$$D = And.$$

EXERCISES

1. Find the simple interest on \$375 at 5% for $2\frac{1}{2}$ years.

2. What sum must be put at interest at 5% for 3 years so that the interest at that time will be \$65?

3. How long will it take \$800 to yield \$96 interest, the rate being 6%?

4. Find the present value and the simple discount on \$500 for 6 months if the rate of interest per year is 5%.

5. Find the simple discount on \$2000 for 5 months at an interest rate of 6% per year. Find the rate of discount.

6. Express d in terms of r and n.

140. **Compound Interest.** If the interest is added to the principal at the end of fixed periods, so that the interest of each succeeding period is computed on this new principal, the amount at the end of a given time is called the *compound amount.* The difference between the compound amount and the original principal is the *compound interest.*

If P is a given sum of money invested at $r\%$ for n years, we have the following compound amounts, at the ends of consecutive years.

$A_1 = P + Pr = P(1 + r)$; at the end of the 1st year.
$A_2 = P(1 + r) + Pr(1 + r) = P(1 + r)^2$; at the end of the 2d year.
$A_3 = P(1 + r)^2 + Pr(1 + r)^2 = P(1 + r)^3$; at the end of the 3d year.

.
.
.

$A_n = P(1 + r)^n$; at the end of the nth year.

Thus the compound amount at the end of the nth year is given by the formula

$$A_n = P(1 + r)^n.$$

It follows that $A_n - P$ is the compound interest.

The time between successive additions of interest is called the *interest period* or the *conversion interval*. Thus for the above formula the interest period is one year. The interest period is usually for a year, a half year, or a quarter year.

141. **Factor of Accumulation.** We have seen that the compound amount may be expressed as

$$A_n = P(1 + r)^n.$$

Thus the compound amount for n years on the principal P is found by multiplying P by $(1 + r)^n$. The factor $(1 + r)^n$ is called the *accumulation factor*. If we let $q = 1 + r$, we have

$$q^n = (1 + r)^n,$$

or, expanding by the binomial theorem, we may write

$$q^n = (1 + r)^n$$
$$= 1 + nr + \frac{n(n-1)}{2!}r^2 + \frac{n(n-1)(n-2)}{3!}r^3 + \cdots.$$

This expansion shows that the compound amount is greater than the simple amount when the time is greater than one year.

142. **General Formula for the Amount.** If the interest is compounded k times per year, the number of interest periods for n years will be nk and the rate per interest period will be r/k where r is the rate per year. Thus the formula becomes

$$A_{k,n} = P\left(1 + \frac{r}{k}\right)^{nk}.$$

For example,

$$A_{2,n} = P\left(1 + \frac{r}{2}\right)^{2n}$$

is the formula where the interest is compounded semi-annually.

EXAMPLE. Find the amount on \$500 for 4 years where the interest is compounded semi-annnally at 4%.

Solution by logarithms.

The formula for the amount is

$$A_{2,4} = P\left(1 + \frac{r}{2}\right)^{8},$$

that is,

$$A_{2,4} = 500(1.02)^{8}.$$

$$\log A_{2,4} = \log 500 + 8 \log 1.02 = 2.6990 + 8(.0086) = 2.7678$$

Hence

$$A_{2,4} = \$585.86$$

Solution by table.

The labor involved in the above solution may be very much reduced by making use of a table which gives the value of $(1 + r)^n$ for various values of r and n. See page 318.

From the table on page 318, the amount on \$1 is \$1.1717

Hence the amount on \$500 is

$$500 \times \$1.1717 = \$585.85$$

143. **Amount—Time Graphs.** In § 138 we found that the graph of the simple amount of a fixed principal at a fixed rate was a straight line. If we let $P = \$1$ and $r = 6\%$, then we have for the simple amount

$$A_n = .06n + 1,$$

and for the compound amount (compounded annually)

$$A_n = (1.06)^n.$$

Computing a table of values for the second formula we have

n =	1	2	5	10	20
Compound Amount =	1.06	1.123	1.338	1.791	3.207

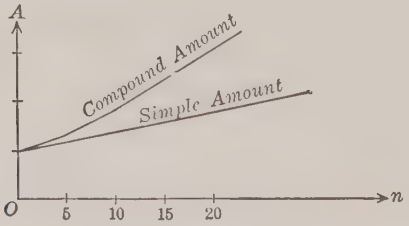


FIG. 35

The graph shows that the compound interest is not proportional to the time as in simple interest, since the graph of this compound amount is not a straight line.

144. **Present Value.—Discount.** The *present value* of an amount A which is due in n years, is the principal P which at compound interest would yield the amount A in n years. If the interest is compounded annually, we find (§ 140)

$$P = A_n(1 + r)^{-n},$$

as the present value.

If the interest is compounded k times per year, we have (§ 142)

$$P = A_{k,n}\left(1 + \frac{r}{k}\right)^{-nk}.$$

If an amount A is due at some future date, the difference between A and its present value is the *discount* on A. Thus $A - P$ is the discount on A.

EXAMPLE. Find the present value of \$1000 due in 3 years if the interest is 5% compounded semiannually.

Since $A = \$1000$, $r/k = .05/2 = .025$, and $nk = 6$, we have $P = \$1000(1 + .025)^{-6}$. Using four-place logarithms, we find

$$\log P = \log 1000 - 6 \log 1.025 = 3.000 - .0642 = 2.9358$$

and $P = \$862.60$; but the last two figures are not reliable on account of interpolation. Using $\log 1.025 = .010723865$, from page 322, we find, to five places, $\log P = 2.93566$, and $P = \$862.30$.

This same value is found directly by using the table on p. 319, where, in the column headed $2\frac{1}{2}\%$, opposite $n = 6$, we find .86230. The discount is \$1000 − \$862.30 = \$137.70

145. **Nominal and Effective Rates.** The rate *per year* at which a sum of money grows is called the *effective rate* as distinguished from the *nominal rate* according to which the principal is compounded. Thus if one dollar is compounded semiannually at 6%, the amount at the end of the first year would be $(1.03)^2 = 1.0609$; that is, the actual gain on one dollar is .0609, and .0609 or 6.09% is the effective rate, while 6% is the nominal rate. It is clear that there is no difference between nominal rate and effective rate if the interest is compounded annually.

To find the formula connecting the two rates we may proceed as follows. Let r = nominal rate and e = effective rate. Then

$$A = \left(1 + \frac{r}{k}\right)^{k},$$

the amount of \$1 compounded k times per year. Hence

$$\left(1+\frac{r}{k}\right)^k - 1 = e,$$

the effective rate. That is,

$$\left(1+\frac{r}{k}\right)^k = 1 + e.$$

EXERCISES

Find the compound amount and the compound interest if

1. \$2000 is invested for 5 years at 5% compounded semiannually.

2. \$3000 is invested for 4 years at 3% compounded quarterly.

3. \$1678 is invested for 12 years at $4\frac{1}{2}$% compounded annually.

Find the present value and the discount on

4. \$5000 due in 3 years if the rate of interest is 5% compounded annually.

5. \$3268 due in $3\frac{1}{4}$ years if the rate of interest is 4% compounded quarterly.

6. \$6500 due in $3\frac{1}{2}$ years compounded semiannually at 5%.

7. Find the effective rate corresponding to 5% compounded quarterly.

8. Find the effective rate corresponding to 8% compounded semiannually.

9. Find by using logarithms the nominal rate, compounded quarterly, that will correspond to an effective rate of 6%.

10. Which is the better investment, 6% compounded semiannually or 6.3% compounded annually?

146. **Annuities.** An *annuity* is a series of equal payments usually made at equal intervals of time. The sum of the payments of an annuity made in one year is called the *annual rent*. The interval of time between successive payments is the *payment interval*. An *annuity certain* is an

annuity that extends over a fixed period of time, that is, from the beginning of the first payment interval to the end of the last payment interval. This fixed period of time is called the *term* of an annuity. Unless otherwise stated, the payments are made at the *end* of each payment interval.

For example, if a man leaves to his son an annuity of $100 per month for a period of 20 years, the payment interval is one month, the annual rent is $1200, and the term of the annuity is twenty years.

A *contingent annuity* is an annuity that extends over a period of time whose length cannot be determined.

For example, if a man leaves $1200 per year to his son, during his son's lifetime, such an annuity is a contingent annuity.

In this chapter we shall consider only annuities certain, and we shall use the term annuity to mean an annuity certain.

147. **Amount of an Annuity.** The amount of an annuity is the value of the annuity at the end of the term of the annuity. Thus we may say that, if the payments of an annuity are allowed to accumulate, the total sum of the payments and interest due at the end of the term is the *amount* of the annuity.

EXAMPLE. Find the amount of an annuity of $1 per annum for 4 years if money is worth 4%.

The first payment is made at the end of the first year and has three years to accumulate; the second payment has two years to accumulate; the third payment one year to accumulate; and the fourth payment does not draw interest. Thus

the compound amount of 1st payment	$= (1.04)^3$	$= \$1.1249$
the compound amount of 2d payment	$= (1.04)^2$	$= \$1.0816$
the compound amount of 3d payment	$= (1.04)$	$= \$1.04$
the amount of 4th payment	$= 1$	$= \$1.$
Total Amount		$= \$4.246$

148. **General Formula for Amount.** The general formula for the amount of an annuity of \$1 per annum for n years at $r\%$ may be found by taking the sum of the geometric progression

$$(1 + r)^{n-1}, (1 + r)^{n-2}, (1 + r)^{n-3}, \cdots, (1 + r), 1.$$

If we reverse the order of the geometric progression, we have

$$\text{Amount} = 1 + (1 + r) + (1 + r)^2 + (1 + r)^3 + \cdots + (1 + r)^{n-1}.$$

We note that the first term of this progression is 1, the common ratio $1 + r$, and that the number of terms is n. Applying the sum formula (§ 37), we have

$$\text{Amount} = \frac{(1 + r)^n - 1}{r}.$$

The amount of an annuity of \$1 per annum for n years is usually denoted by the symbol $s_{n\rceil}$. Thus

$$s_{n\rceil} = \frac{(1 + r)^n - 1}{r}.$$

The values of $S_{n\rceil}$ for various values of n and r have been computed and can be found in printed tables. See page 320.

If in the example of § 147 we use this formula, we have

$$s_{4\rceil} = \frac{(1.04)^4 - 1}{.04} = \frac{.1699}{.04} = \$4.2465$$

If the annual payments are P dollars per annum, the amount is

$$S_{n\rceil} = P\frac{(1 + r)^n - 1}{r}.$$

Thus in the example above, the amount of the annuity of \$100 per annum instead of \$1 per annum would be $100 \times 4.2465 = \$424.646$

EXERCISES

1. What will be the amount of the accumulated value of an annuity of $50 per year for 10 years if money is worth 4%?

2. Find the amount of an annuity of $200 per year for 20 years if the current rate of interest is 5%.

3. What is the amount of an annuity of $134.50 per year for 8 years where the current rate of interest is 3%?

4. Show that the amount of an annuity of $1, payable m times a year for n years, the interest being compounded at each payment, is

$$\frac{\left(1+\frac{r}{m}\right)^{mn}-1}{r/m}.$$

5. Find the amount of an annuity if $250 is payable each half-year for 6 years, and if the current rate of interest is 4%.

6. Find the amount of an annuity if $150 is payable each quarter for 10 years and 6 months, and if the current rate of interest is 5%.

149. **Present Value of an Annuity.** The *present value* of an annuity is the sum of the present values of all of the payments. Thus the present value of an annuity is the cash equivalent of the annuity.

EXAMPLE. Find the present value of an annuity of $1 per annum for 4 years if money is worth 4%.

In finding the present value we see that in this example the first payment of $1 must be discounted for one year, the second payment for two years and the third payment for three years and the fourth and last payment for four years. Thus by § 144,

the present value of $1 discounted for 1 year $= (1.04)^{-1} = .96154$
the present value of $1 discounted for 2 years $= (1.04)^{-2} = .92456$
the present value of $1 discounted for 3 years $= (1.04)^{-3} = .8890$
the present value of $1 discounted for 4 years $= (1.04)^{-4} = .85480$

$$\text{Present value} = (1.04)^{-1} + (1.04)^{-2} + (1.04)^{-3} + (1.04)^{-4}$$
$$= \$3.6299 = \$3.63$$

150. **General Formula for Present Value.** The general formula for the present value of \$1 per annum for n years at $r\%$ may be found by taking the sum of the geometric progression

$$(1 + r)^{-1}, (1 + r)^{-2}, (1 + r)^{-3}, \cdots, (1 + r)^{-n}.$$

That is,

$$\text{Present value} = (1 + r)^{-1} + (1 + r)^{-2} + (1 + r)^{-3} + \cdots + (1 + r)^{-n}.$$

The first term in the progression is $(1 + r)^{-1}$, the common ratio is $(1 + r)^{-1}$ and n is the number of terms. Applying the sum formula (§ 37), we have

$$\text{Present value} = \frac{(1 + r)^{-1}[1 - (1 + r)^{-n}]}{1 - (1 + r)^{-1}} = \frac{1 - (1 + r)^{-n}}{r}.$$

Thus we have for the present value of an annuity of \$1 per annum for n years at $r\%$ the formula

$$a_{\overline{n}|} = \frac{1 - (1 + r)^{-n}}{r}.$$

If the annual payments are P dollars per annum, then the present value of the annuity is

$$A_{\overline{n}|} = P\frac{1 - (1 + r)^{-n}}{r}.$$

Here again one can save much labor by making use of a table (page 319) which gives the value of $(1 + r)^{-n}$ for various values of r and n, or still better, a table (page 321) which gives the values of $a_{\overline{n}|}$.

EXERCISES

1. Find the present value of an annuity of \$200 per annum for 5 years if the current interest rate is 5%.

2. Find the cash equivalent of an annuity of \$150 per annum for 10 years if $r = .04$.

3. Find the amount and also the present value of an annuity of \$324 per annum for 8 years if the current rate of interest is $3\frac{1}{2}\%$.

4. Show that

$$\frac{1 - \left(1 + \frac{r}{m}\right)^{-mn}}{r/m}$$

is the formula for the present value of an annuity of \$1 payable m times a year for n years, the interest being compounded at each payment.

5. Show that

$$P = \frac{rS_{\overline{n}|}}{(1 + r)^n - 1}.$$

6. Using the formula in Ex. 5 find P if $S_{\overline{n}|} = \$2000$, $n = 6$ and $r = .05$

7. What is the annual rent of an annuity for 8 years if the amount is \$20000 and the rate is 5%?

8. Show that

$$P = \frac{rA_{\overline{n}|}}{1 - (1 + r)^{-n}}.$$

9. In Ex. 8 find P if $A_{\overline{n}|} = \$1000$, $n = 4$ and $r = .06$

10. Find the annual rent of an annuity for 5 years whose present value is \$6000 if money is worth 4%.

11. A man buys some property making a cash payment of \$2000 and agrees to pay \$500 annually for the next 6 years. If money is worth 5%, what is the equivalent cash price?

12. A man rents property at an annual rental of \$2000, agreeing to pay his rent quarterly in advance. He wishes to pay two years' rent in advance. How much should he pay if money is worth 6%?

CHAPTER XVII

COMPLEX NUMBERS

151. **Standard Form $a + bi$.** In § 16 we defined a complex number as the algebraic sum of a real number and a pure imaginary number, that is, the standard form of a complex number is

$$a + bi,$$

where a and b are real and $i = \sqrt{-1}$.

Furthermore we found that if $i = \sqrt{-1}$, then

$$i^2 = -1, \qquad i^3 = -i, \qquad i^4 = +1,$$

and that any positive integral power of i is equal to one of the four symbols, $+1$, -1, $+i$, or $-i$.

For example, $i^7 = i^4 \cdot i^3 = +1 \cdot -i = -i$.

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. That is,

$$a + bi = c + di,$$

if and only if $a = c$ and $b = d$. From this it follows that if $a + bi = 0$, then $a = 0$ and $b = 0$.

If two complex numbers differ only in the sign of their imaginary parts, they are called *conjugate complex numbers*. Thus $a + bi$ and $a - bi$ are conjugate.

We note that

$$a + bi + a - bi = 2a, \text{ a real number,}$$

and

$$(a + bi)(a - bi) = a^2 + b^2, \text{ a real number.}$$

Thus the sum and the product of two conjugate complex numbers are real numbers.

152. **The Four Fundamental Operations.** It is beyond the scope of this book to give a complete treatment of the algebra of complex numbers. We shall assume without proof that the imaginary unit i is subject to the laws of algebra applying to real numbers. Furthermore each integral power of i should be expressed as one of the four symbols $+1$, -1, $+i$, or $-i$, as in § 151.

Addition. $a + bi + c + di = (a + c) + (b + d)i.$

Multiplication.
$$\begin{aligned}(a + bi)(c + di) &= ac + (bc + ad)i + bdi^2 \\ &= (ac - bd) + (bc + ad)i.\end{aligned}$$

Division.
$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} \cdot i.\end{aligned}$$

Thus in division we multiply both numerator and denominator by the conjugate of the denominator.

EXAMPLE 1. Express $(1 + i)^3$ in the form of $a + bi$.

$$(1 + i)^3 = 1 + 3i + 3i^2 + i^3 = 1 + 3i - 3 - i = -2 + 2i.$$

EXAMPLE 2. Divide $5 + 6i$ by $3 + 2i$.

$$\frac{5 + 6i}{3 + 2i} = \frac{(5 + 6i)(3 - 2i)}{(3 + 2i)(3 - 2i)},$$

$$= \frac{15 + 18i - 10i - 12i^2}{9 - 4i^2} = \frac{27 + 8i}{13} = \frac{27}{13} + \frac{8i}{13}.$$

EXERCISES

Reduce to simplest form.

1. i^6.
2. i^5.
3. i^7.
4. i^8.
5. $i^2 + 2i$.
6. $i^3 + 3i^2$.
7. $i^4 + 2i$.
8. $i^4 + i^8$.
9. $(i + 1)^2$.
10. $i^2 - 1$.
11. $i^2 + 1$.
12. $i^3 + 1$.

Perform the indicated operations and reduce to the form of $a + bi$.

13. $(2 + i) + (3 + 2i)$.
14. $(6 + \sqrt{-4}) + (3 - \sqrt{-2})$.
15. $(3 - \sqrt{-5}) + (2 + \sqrt{-4})$.
16. $(\sqrt{4} - \sqrt{-9}) + (2 + \sqrt{-4})$.
17. $(3i + 6) + (7 - 4i)$.
18. $(2 + 5\sqrt{-3}) + (3 + 6\sqrt{-3})$.

Find the product of each of the following and express the result in the form $a + bi$.

19. $(3 + 6i)(4 - 5i)$.
20. $(i + 1)(1 - i)(2 + 3i)$.
21. $(2 + 3i)^2(1 + i)^2$.
22. $(2 - i)^2(2 + i)^2$.

23. Find the product of $2 + 3i$ and its conjugate.

24. Find the sum of $2 + 3i$ and its conjugate.

Reduce the following quotients to the form of $a + bi$.

25. $\dfrac{6 + 3i}{2 - i}$.
26. $\dfrac{5 - 3i}{4 + i}$.
27. $\dfrac{2 + 3i}{3 - 4i}$.
28. $\dfrac{2 + \sqrt{-4}}{3 + 2\sqrt{-3}}$.
29. $\dfrac{5 - \sqrt{-2}}{2 + \sqrt{-3}}$.
30. $\dfrac{1}{2 + \sqrt{-5}}$.

Find the values of x and y where

31. $x + yi = 6 + 7i$.
32. $2x - 3yi = (1 + i)^2$.
33. $3x + iy - 4i + 6 = 0$.
34. $x - iy - (1 + i)^3 = 0$.

153. **The Polar Form of Complex Numbers.** We have seen (§ 60) that a complex number $a + bi$ can be represented by a

vector. Another convenient method of representing complex numbers is the polar or the trigonometric method.

In Fig. 36 let $\theta = \angle XOP$ and $r = OP$. We shall assume that θ is positive when measured counter clock-wise and negative when measured clock-wise.

We now have

$$a = r\cos\theta,$$
$$b = r\sin\theta.$$

Hence

$$\boldsymbol{a + bi = r(\cos\theta + i\sin\theta).}$$

The form $r(\cos\theta + i\sin\theta)$ is called the *polar form* of the complex number $a + bi$.

The angle θ is the *angle* or the *argument*, and r is the *absolute value* or the *modulus* of the complex number (§ 60).

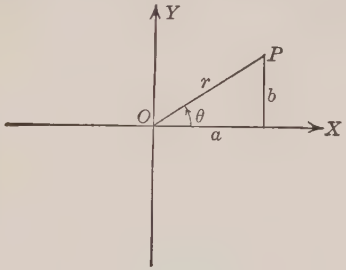


FIG. 36

154. **From Rectangular to Polar Form.** The rectangular form of a complex number is $a + bi$. To express $a + bi$ in the polar form we need to know r (the absolute value) and θ (the angle).

Since $r\cos\theta = a$ and $r\sin\theta = b$, we have

$$r = \sqrt{a^2 + b^2}, \qquad \text{and} \qquad \tan\theta = \frac{b}{a},$$

from which θ may be found.

We observe that, since $\tan(\theta + k\cdot 360°) = \tan\theta$, where k is an integer, there are an indefinite number of permissible angles. It is customary to select the smallest positive or negative angle.

EXAMPLE. Find the polar form of $1 + \sqrt{3}i$.

Let $1 + \sqrt{3}i = a + bi$.

Then $a = 1$, $b = \sqrt{3}$, and

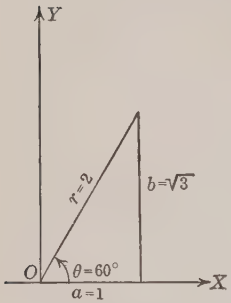


FIG. 37

$$r = \sqrt{1 + 3} = 2, \qquad \tan\theta = \sqrt{3}.$$

Hence $\theta = 60°$, and we may write

$$1 + \sqrt{3}i = 2(\cos 60° + i \sin 60°).$$

155. **From Polar to Rectangular Form.** To pass from the polar to the rectangular form of a complex number we need to know the values of $\cos\theta$ and $\sin\theta$. These can be found from the table on page 299.

EXAMPLE. Reduce $2(\cos 55° + i \sin 55°)$ to the form of $a + bi$.

From the table,

$$\cos 55° = .5736, \qquad \text{and} \qquad \sin 55° = .8192$$

Hence

$$2(\cos 55° + i \sin 55°) = 2(.5736 + .8192i) = 1.1472 + 1.6384i.$$

EXERCISES

Reduce to polar form

1. $1 - \sqrt{3}i$.	**5.** i.	**9.** $3 + 4i$.
2. $3 + 3i$.	**6.** $2 + 2i$.	**10.** $12 + 5i$.
3. $\sqrt{3} - i$.	**7.** $-3i$.	**11.** $6 + 7i$.
4. $\sqrt{3} + i$.	**8.** $4 + i$.	**12.** $3 - 8i$.

Plot each of the following complex numbers and reduce to the form of $a + bi$.

13. $2(\cos 30° + i \sin 30°)$.	**16.** $3(\cos 90° + i \sin 90°)$.
14. $3(\cos 120° + i \sin 120°)$.	**17.** $7(\cos 180° + i \sin 180°)$.
15. $5(\cos 105° + i \sin 105°)$.	**18.** $5(\cos 270° + i \sin 270°)$.

156. **Polar Form Products and Quotients.** We shall now establish a simple rule for finding the product of two complex numbers given in the polar form. We have at once

$$\begin{aligned} r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2[(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)] \\ &= r_1r_2[\cos\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2 \\ &\qquad + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \\ &\qquad + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)] \\ &= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \end{aligned}$$

Thus we have: *The absolute value of the product of two complex numbers is equal to the product of their absolute values, and the angle of the product is equal to the sum of their angles.*

For example:

$$\begin{aligned} 2(\cos 30^\circ + i\sin 30^\circ)\cdot 3(\cos 40^\circ + i\sin 40^\circ) \\ &= 6(\cos 70^\circ + i\sin 70^\circ). \end{aligned}$$

Similarly we may express the quotient of two complex numbers in the form

$$\begin{aligned} \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} &= \frac{r_1}{r_2}\left[\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2}\cdot\frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}\right] \\ &= \frac{r_1}{r_2}\left[\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)}{\cos^2\theta_2 + \sin^2\theta_2}\right], \\ &= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]. \end{aligned}$$

Thus we have: *The absolute value of the quotient of two complex numbers is equal to the quotient of the absolute values and the angle of the quotient is equal to the angle of the dividend minus the angle of the divisor.*

For example:

$$\frac{8(\cos 75^\circ + i\sin 75^\circ)}{2(\cos 30^\circ + i\sin 30^\circ)} = 4(\cos 45^\circ + i\sin 45^\circ).$$

EXERCISES

Find in polar form the indicated products and quotients in the exercises below.

1. $2(\cos 30° + i \sin 30°) \cdot 5(\cos 60° + i \sin 60°)$.
2. $\sqrt{2}(\cos 15° + i \sin 15°) \cdot \sqrt{3}(\cos 30° + i \sin 30°)$.
3. $3(\cos 95° + i \sin 95°) \cdot 4(\cos 76° + i \sin 76°)$.
4. $\dfrac{2(\cos 80° + i \sin 80°)}{3(\cos 35° + i \sin 35°)}$.
5. $\dfrac{\cos 135° + i \sin 135°}{\cos 90° + i \sin 90°}$.
6. $\dfrac{2(\cos 240° + i \sin 240°)}{3(\cos 150° + i \sin 150°)}$.
7. $\dfrac{8(\cos 60° + i \sin 60°)}{4(\cos 240° + i \sin 240°)}$.
8. Express in polar form and then find the product of $(2 + 2i)$ and $(1 - \sqrt{3}i)$.
9. Express in polar form and divide $(2 + 2i)$ by $(1 - \sqrt{3}i)$.

157. **DeMoivre's Theorem.** In § 156 we found that the product of two complex numbers could be simply expressed as

$$r_1(\cos \theta_1 + i \sin \theta_1) \cdot r_2(\cos \theta_2 + i \sin \theta_2) = r_1 r_2[\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)].$$

If $\theta_1 = \theta_2$, and $r_1 = r_2$, the above expression becomes

$$[r_1(\cos \theta_1 + i \sin \theta_1)]^2 = r_1^2(\cos 2\theta_1 + i \sin 2\theta_1).$$

In fact, it can be proved by mathematical induction (§ 52), where n is a positive integer, that

$$[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta).$$

This formula expresses DeMoivre's theorem, which may be stated as follows: *The absolute value of the nth power of a complex number is equal to the nth power of the absolute value and the angle of the nth power is equal to n times the angle of the number.*

For example:

$$[2(\cos 30° + i \sin 30°)]^4 = 2^4(\cos 4 \cdot 30° + i \sin 4 \cdot 30°) = 16(\cos 120° + i \sin 120°).$$

EXERCISES

Making use of DeMoivre's theorem find the indicated powers of the following complex numbers.

1. $[3(\cos 45° + i \sin 45°)]^2$.
2. $[\sqrt{2}(\cos 60° + i \sin 60°)]^4$.
3. $[2(\cos 20° + i \sin 20°)]^5$.
4. $[4(\cos 15° + i \sin 15°)]^4$.
5. $(1 + i)^6$.
6. $(1 - \sqrt{3}i)^4$.
7. $(2 - 2i)^3$.
8. $(\sqrt{3} + i)^2$.
9. Prove DeMoivre's theorem by the method of mathematical induction, as in Chapter VI.

158. **Roots of a Complex Number.** We have seen how DeMoivre's theorem can be used in finding the powers of a complex number. We shall now illustrate the application of the theorem in finding the roots of a complex number.

Example 1. Find the square roots of $4(\cos 120° + i \sin 120°)$.

Let $\sqrt{4(\cos 120° + i \sin 120°)} = r(\cos \theta + i \sin \theta)$. Squaring both sides, we have

$$4(\cos 120° + i \sin 120°) = r^2(\cos 2\theta + i \sin 2\theta).$$

Whence $r = 2$ and $2\theta = 120° + k \cdot 360°$, where k is a positive integer, or

$$r = 2, \qquad \text{and} \qquad \theta = \frac{120° + k \cdot 360°}{2}.$$

If $k = 0$, $\theta = 60°$; if $k = 1$, $\theta = 240°$. Hence the two square roots are

$$\sqrt{4(\cos 120° + i \sin 120°)} = 2(\cos 60° + i \sin 60°),$$

and

$$\sqrt{4(\cos 120° + i \sin 120°)} = 2(\cos 240° + i \sin 240°).$$

Example 2. Find the cube roots of $27(\cos 60 + i \sin 60)$.

Let

$$\sqrt[3]{27(\cos 60° + i \sin 60°)} = r(\cos \theta + i \sin \theta).$$

Raising both sides to the third power, we have

$$27(\cos 60° + i \sin 60°) = r^3(\cos 3\theta + i \sin 3\theta).$$

Whence $r = 3$, and $3\theta = 60° + k\cdot 360°$, or

$$r = 3, \qquad \text{and} \qquad \theta = \frac{60° + k\cdot 360°}{3}.$$

For $k = 0$, $\theta = 20°$; for $k = 1$, $\theta = 140°$; for $k = 2$, $\theta = 260°$.

Hence the three cube roots of $27(\cos 60° + i \sin 60°)$ are

$$\sqrt[3]{27(\cos 60° + i \sin 60°)} = 3(\cos 20° + i \sin 20°),$$
$$\sqrt[3]{27(\cos 60° + i \sin 60°)} = 3(\cos 140° + i \sin 140°),$$
$$\sqrt[3]{27(\cos 60° + i \sin 60°)} = 3(\cos 260° + i \sin 260°).$$

If we represent these three cube roots graphically, we observe that the end of each vector lies on a circle whose radius is the *absolute value* of the roots. Furthermore the three points P, Q and R divide the circumference into three equal arcs.

159. **General Statement.** If we follow the method indicated in the illustrative examples of § 158, we find that every complex number $r(\cos\theta + i \sin\theta)$ has n distinct nth roots, that is,

$$\sqrt[n]{r(\cos\theta + i\sin\theta)} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + k\cdot 360}{n}\right) + i\sin\left(\frac{\theta + k\cdot 360}{n}\right)\right],$$

where k takes on the values of 0, 1, 2, $\cdots$, $n - 1$, and where $\sqrt[n]{r}$ means the real positive nth root of r.

EXERCISES

By the method used in Examples 1 and 2 (§ 158), find the indicated roots and represent them graphically.

1. $\sqrt[3]{8(\cos 30° + i \sin 30°)}$.
2. $\sqrt[4]{16(\cos 240° + i \sin 240°)}$.
3. $\sqrt[5]{32(\cos 45° + i \sin 45°)}$.
4. $\sqrt{9(\cos 20° + i \sin 20°)}$.
5. Express $(8\sqrt{2} + 8\sqrt{2}i)$ in polar form and find the 4th roots.
6. Find the cube roots of $\dfrac{1 - \sqrt{3}i}{2}$.

7. Find the square roots of i.

8. Find the cube roots of 1.

9. Find the fourth roots of -81.

Solve the following equations and represent the roots graphically.

10. $x^3 - 1 = 0$.

11. $x^3 + 1 = 0$.

12. $x^5 - 32 = 0$.

13. $x^6 - 1 = 0$.

14. $x^3 + 125 = 0$.

15. $x^4 - 16 = 0$.

CHAPTER XVIII

SCALES OF NOTATION

160. **Common Scale.** Ordinarily the numbers which we use in mathematics are expressed by means of powers of ten, that is, they are written in the *decimal* or the *common* scale. Thus

$$324 = 3\cdot 10^2 + 2\cdot 10 + 4$$

and

$$4231 = 4\cdot 10^3 + 2\cdot 10^2 + 3\cdot 10 + 1.$$

It is evident that 10 plays an important role in expressing numbers in this way. Thus 10 is called the *base* of the decimal or the common scale. In the same way we may select any number as a base. For example if the base is 6, any number as 324 is to be interpreted as follows:

$$324 \text{ (scale 6)} = 3\cdot 6^2 + 2\cdot 6 + 4,$$

and

$$4231 \text{ (scale 6)} = 4\cdot 6^3 + 2\cdot 6^2 + 3\cdot 6 + 1.$$

161. **General Theorem.** If r is taken as a base of a scale of notation and N is any positive integer, then

$$N = a_0 r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_n,$$

where a_0, a_1, a_2, $\cdots$, a_n are positive integers less than r.

Proof. Let r^n be the highest power of r contained in N. Then, if a_0 is the quotient and R_1 the remainder, obtained by dividing N by r^n, we have

$$N = a_0 r^n + R_1.$$

Dividing R_1 by r^{n-1}, we have

$$R_1 = a_1 r^{n-1} + R_2.$$

Continuing this process, we find

$$\begin{aligned} R_2 &= a_2 r^{n-2} + R_3, \\ &\cdots \\ R_{n-1} &= a_{n-1} r + a_n. \end{aligned}$$

Substituting in the first equation the value of R_1 from the second, we have

$$N = a_0 r^n + a_1 r^{n-1} + R_2.$$

If we continue the substitutions, for R_2, R_3, $\cdots$, R_{n-1}, we have finally

$$N = a_0 r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_n.$$

We observe in the above proof that each integer a_0, a_1, a_2, $\cdots$, a_n is less than r. Furthermore, some of them may be zero, and finally it is evident that, including zero, r digits are required to express a number in the scale of r.

162. **Changing Scales.** As integral numbers may be expressed in any scale, we shall need to convert a number in a given scale r_1, into a number in any other scale r_2. We may divide our problem into three parts.

(1) To change from any scale r to the scale 10.
(2) To change from the scale 10 to any scale r.
(3) To change from the scale r_1 to any other scale r_2.

163. **From Any Scale r to the Scale of 10.** The general theorem (§ 161) gives us a method for converting a number from a given scale r into a number in the common scale. Thus

$$N = a_0 r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_n$$

is the formula to be used.

EXAMPLE. Express 236 (scale 7) in the common scale.

By definition

$$236 \text{ (scale 7) is } 2\cdot 7^2 + 3\cdot 7 + 6.$$

Hence

$$N \text{ (scale 10)} = 2\cdot 7^2 + 3\cdot 7 + 6 = 125.$$

That is,

$$236 \text{ (scale 7)} = 125 \text{ (scale 10)}.$$

164. **From the Scale of 10 to Any Scale r.** If N be a given number (scale 10) and r the base of the scale required, we have

$$N = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n,$$

where $a_0, a_1, \cdots, a_n$ are to be determined. If N be divided by r, we have

$$\frac{N}{r} = Q_1 + \frac{a_n}{r},$$

that is, a_n is the remainder.

If the quotient Q_1 be divided by r, the second remainder is a_{n-1}. By continuing this process we shall find all of the integers $a_0, a_1, a_2, \cdots, a_n$.

EXAMPLE. Express 236 (scale 10) in the scale of 5.

We have

$$236 = a_0\cdot 5^3 + a_1\cdot 5^2 + a_2\cdot 5 + a_3,$$

where a_0, a_1, a_2, and a_3 are to be determined. Thus

$$\begin{array}{r|l} 5 & 236 \\ 5 & 47, \text{ remainder } 1 = a_3 \\ 5 & 9, \text{ remainder } 2 = a_2 \\ & 1, \text{ remainder } 4 = a_1. \end{array}$$

Hence we have the number 1421. That is,

$$236 \text{ (scale 10)} = 1421 \text{ (scale 5)}.$$

165. **From Any Scale r_1 to Any Other Scale r_2.** In order to convert a number in a scale r_1 to a number in a scale r_2, it is convenient to convert the number (scale r_1) into the corresponding number (scale 10) and then convert the latter into the corresponding number (scale r_2).

EXAMPLE. Convert 261 (scale 7) into the corresponding number (scale 6).

Since

$$261 \text{ (scale 7)} = 2 \cdot 7^2 + 6 \cdot 7 + 1 = 141 \text{ (scale 10)},$$

we have now to convert 141 (scale 10) into a number (scale 6). Thus

$$\begin{array}{r|l} 6 & 141 \\ 6 & 23, \text{ remainder } 3 \\ & 3, \text{ remainder } 5. \end{array}$$

That is, 141 (scale 10) = 353 (scale 6). Hence

$$261 \text{ (scale 7)} = 141 \text{ (scale 10)} = 353 \text{ (scale 6)}.$$

EXERCISES

Change each of the numbers from the scale indicated to the common or decimal scale.

1. 313 (scale 5).

2. 1647 (scale 9).

3. 1210 (scale 3).

4. 2011 (scale 8).

Convert the following numbers as indicated:

5. 3678 (scale 10) = what number (scale 9).

6. 1211 (scale 10) = ? (scale 6).

7. 3267 (scale 10) = ? (scale 7).

8. 2126 (scale 7) = ? (scale 9).

9. 3624 (scale 8) = ? (scale 5).

10. 1111 (scale 9) = ? (scale 7).

166. **The Four Fundamental Operations.** Addition, subtraction, multiplication, and division can be performed in any scale in much the same way as in the common scale, the fundamental difference being that in the common or decimal scale we carry and borrow tens, while in any other scale we carry and borrow the number of units indicated by the scale in which we are working.

EXAMPLE 1. Add 124 and 245, scale 6.

In performing addition it may prove helpful at first to write the numbers in the expanded form. Thus

$$\begin{aligned} 124 &= 1\cdot 6^2 + 2\cdot 6 + 4 \\ 245 &= 2\cdot 6^2 + 4\cdot 6 + 5 \\ \hline 413 &= 4\cdot 6^2 + 1\cdot 6 + 3. \end{aligned}$$

Adding the last column, $5 + 4 = 9 = 1\cdot 6 + 3$, hence, we write down 3 and carry the 1 to the second column. Adding the second column, $1 + 4 + 2 = 7 = 1\cdot 6 + 1$, hence, we put down 1 and carry 1 to the first column. Adding the first column, $1 + 2 + 1 = 4$. Thus the sum is 413.

EXAMPLE 2. From 2313 subtract 1635, scale 7.

$$\begin{aligned} 2313 &= 2\cdot 7^3 + 3\cdot 7^2 + 1\cdot 7 + 3 \\ 1635 &= 1\cdot 7^3 + 6\cdot 7^2 + 3\cdot 7 + 5 \\ \hline 345 &= \qquad\quad 3\cdot 7^2 + 4\cdot 7 + 5. \end{aligned}$$

Since we cannot take 5 from 3 we borrow 1, remembering that it is one *seven*. Thus $3 + 7 = 10$ and $10 - 5 = 5$.

EXAMPLE 3. Find the product of 324 and 432 (scale 5).

$$\begin{array}{r} 324 \\ 432 \\ \hline 1203 \\ 2032 \\ 2411 \\ \hline 313123 \end{array}$$

In the process of multiplication we must remember that we are carrying *fives*. Thus we have $2 \times 4 = 8 = 1\cdot 5 + 3$, so we put down the 3 and carry the 1. Next $2 \times 2 = 4$, and adding the 1 that was carried we have $1 + 4 = 5 = 1\cdot 5 + 0$, so we put down the 0 and carry the 1.

Example 4. Divide 14064 by 25, scale 7.

25 | 14064 | 405
136
164
164

In estimating the first figure in the quotient we must keep the scale carefully in mind. Thus

$$2 \times 4 = 8 = 1 \cdot 7 + 1 = 11 \text{ (scale 7)}.$$

Hence it is evident that the first figure in the quotient is 4.

EXERCISES

Perform the operations indicated.

1. 2312 + 6734 + 1368, scale 9.
2. 1203 + 2323 + 1312, scale 4.
3. 5673 − 2346, scale 8.
4. 13262 − 6236, scale 7.
5. 262 × 324, scale 7.
6. 1234 × 242, scale 5.
7. 1224 ÷ 223, scale 5.
8. 6072 ÷ 234, scale 9.

ANSWERS

CHAPTER I

Pages 5–6

1. $12x + 6y - 15z$. **2.** $15x^3 - 5x^2 + 4x + 4$. **3.** $7m^3 - 5m^2 - 1$.
4. $a^3 + 3a^2b - 3ab^2 + b^3$. **5.** $9(a + b) - 9(b + c) - 22(c + a)$.
6. $14(a + b - c) + 3(x - y) + 6(a - b)$. **7.** $x^4 - 4ax^3 + 6a^2x^2 - 4a^3x$.
8. $3x^3 + 10x^2y - 10xy^2$. **9.** $(b - c)x + (c - a)y + (a - b)z$.
10. $a - 2b$. **11.** 0. **12.** 0. **13.** $3a$. **14.** $2m + 14n - 7p$.
15. $(a - b + 3)x + (b - c + 5)y + (c - 4)z$. **18.** $2a - 6b - 3c$.
19. $5xy + 8y^2$. **20.** $2x + 2y + 2z$. **21.** $4x$. **22.** $4y$. **23.** $4z$.
24. $2x^2 - xy - 3y^2$. **25.** $4a^2 - 2a - 72$. **26.** $a^2b^2 + abcd - 2c^2d^2$.
27. $2x^4 - 3x^3 + 3x^2 - x$. **28.** $2a^4 + 3a^3b - 3a^2b^2 + 3ab^3 - b^4$.
29. $x^6 + 10x - 33$. **30.** $a^3 + b^3 + 3ab - 1$. **31.** $x + 2$. **32.** $2a - 1$.
33. $x - 1$. **34.** $a^2 - 2a + 4$. **35.** $x^4 - 5x^2 + 4$.
36. $a^2 - ab + ac + b^2 + bc + c^2$. **37.** $x^3 + 2x^2y - xy^2 - 2y^3$.

Pages 7–8

7. $9x^2 - 3x - 56$. **9.** $4a^4 + 4a^3b - 11a^2b^2 - 6ab^3 + 9b^4$.
10. $(m + n)^3 + 3(m + n)^2p + 3(m + n)p^2 + p^3$. **11.** $256 - x^8$.
12. $x^4 - y^4 - 2y^2z^2 - z^4$. **13.** $2p(3m^2 + 6mn + 3n^2 + p^2)$.
14. $4(a^2 + b^2 + c^2)$. **15.** $8b^3$. **16.** $4x^2 + 2x + 1$. **17.** $9a^2 + 3ab + b^2$.
18. $6x^2 - 7$. **19.** $2 + a + b$. **20.** $1 + 2x + 4x^2 + 8x^3$. **21.** $a^8 - 3a^4 + 9$.
22. $x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$. **23.** $x^{2m} - 2x^m + 4$.

Page 11

1. $x - 4$. **2.** $3x + 5$. **3.** $4a - 3$. **4.** $a^2 + a - 6$. **5.** $2x^2 + 2$, rem.9.
6. $x + 7a$. **7.** $a^3 - a^2 + a - 1$. **8.** $a^5 + a^4 + a^3 + a^2 + a + 1$.
9. $x^3 - 4x^2 - 6x - 2$. **10.** $2x^4 - x^3 + 2x - 1$, rem.5. **11.** $x^2 + 2x - 4$.

Pages 14–15

9. $(2x - 7)(4x + 3)$. **14.** $(x - 7y)(x - 13y)$. **19.** $(4a - 3b)(5a + 6b)$.
25. $5xy(x - y)(2x + y)$. **30.** $(a + b)(a - b)(a^2 + ab + b^2)(a^2 - ab + b^2)$.
32. $(a - b)(a + b)(a^2 + b^2)(a^4 + b^4)$.
34. $(a - b)(a + b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$.
36. $(x + y - z)(x^2 + 2xy + y^2 + xz + yz + z^2)$. **44.** $(2a + b)(2a + b)$.
46. $(x + 2y - z - 2p)(x^2 + 4xy + 4y^2 + xz + 2yz + 2xp + 4yp + z^2 + 4zp + 4p^2)$.
48. $(9x + 7y + mn - 8a^2)(9x + 7y - mn + 8a^2)$.
49. $(3a^2 + 5ax + 5x^2)(3a^2 - 5ax + 5x^2)$.

Pages 16–17

1. $\frac{5b^2}{3ac^3}$. **2.** $\frac{5xy}{2}$. **3.** $\frac{a+2}{a+5}$. **4.** $\frac{2a+5}{2a-5}$. **5.** $\frac{5+x}{6-x}$.

6. $\frac{a+4}{3a-4}$. **7.** $\frac{5x}{2y^3}$. **8.** $\frac{x^2+2x+4}{x}$. **9.** $\frac{1}{m^4+m^2+1}$.

10. $\frac{m^3+n}{m^3+3n}$. **11.** $\frac{a-3b+c}{a-3b-c}$. **12.** $\frac{a^2+4ab+2b^2}{2a+5b}$.

Pages 18–19

1. $\frac{9a}{10}$. **2.** $\frac{33+4a^2b-30b^3}{6ab^2}$. **3.** $\frac{a+1}{10}$. **4.** $\frac{7x-62}{60}$. **5.** $\frac{3a+9b}{8a}$.

6. $\frac{a^3+b^3+c^3-3abc}{abc}$. **7.** $\frac{a^4b^2+a^2c^4-b^4c^2}{a^2b^2c^2}$. **8.** $-\frac{a+7b+5c}{28}$.

9. $\frac{2x+12}{(x-6)(x+2)}$. **10.** $\frac{a^2}{(x-2a)(x-a)}$. **11.** $\frac{24xy}{4x^2-9y^2}$.

12. $\frac{3x-5}{(x-2)(x-1)}$. **13.** $\frac{a^2}{a-x}$. **14.** $\frac{4x-5y}{16x^2-y^2}$. **15.** $\frac{1}{a-1}$.

16. $\frac{2}{1+a}$. **17.** $\frac{23}{a^2-7a}$. **18.** $\frac{2(a^2-ab+b^2)}{a(a-b)^2}$.

19. $\frac{2}{(a-4)(a-6)}$. **20.** $\frac{2a^3}{a^2-b^2}$. **21.** $\frac{a^2+ab+2b^2}{a^2-b^2}$.

22. $\frac{2}{x(1-4x^2)}$. **23.** $\frac{9a^3}{2a-1}$. **24.** $\frac{6}{(r-1)(r+1)(r+3)}$.

25. $\frac{3a-11}{(a-1)(a-2)(a-3)}$. **26.** $\frac{2(17a^2-38)}{(a^2-1)(a^2-4)}$. **27.** $\frac{2}{(2a+1)(a+2)(4a-3)}$.

28. $\frac{x^5}{1-x^8}$. **29.** $\frac{2a(a+b)}{a^3+8b^3}$. **30.** $\frac{36x^4}{x^8-6561}$. **31.** 1. **32.** 0. **33.** 0.

Pages 20–21

1. $\frac{a}{bc^3}$. **2.** $\frac{4xy^2}{3z}$. **3.** $\frac{7pq}{5r^2}$. **4.** $\frac{1}{3x^3y}$. **5.** $\frac{1}{2}$. **6.** $\frac{4x+3a}{x+2}$.

7. $\frac{(b^2+4)(2-b)}{(1-b)^2}$. **8.** $\frac{6}{a+2}$. **9.** $\frac{m-2}{m+1}$. **10.** $\frac{a}{(a-b)(a-7b)}$.

11. $\frac{x^2y^2}{x^2-y^2}$. **12.** $\frac{x-3}{x+2}$. **13.** $-\frac{1}{a^2}$. **14.** $-\frac{1}{2}$.

15. $\frac{(a^2+b^2)(a^2+ab+b^2)}{(a-b)^2(a^2-ab+b^2)}$. **16.** $\frac{m+n+p}{m-n-p}$. **17.** $\frac{10a(a+10)}{(a+1)(a-8)}$.

18. $a + b$. **19.** $(x + 1)(x + 2)(x - 6)$. **20.** $\frac{a^2 + ab - b^2}{2a^2b}$. **21.** $\frac{1}{b}$.
22. $\frac{(c - 3d)(a - 5b)}{(3c + d)(5a + b)}$.

Pages 24–26

1. 4. **2.** $\frac{27}{8}$. **3.** 3. **4.** -2. **5.** -40. **6.** 3. **7.** 4. **8.** $\frac{38}{33}$.
9. $\frac{2ab}{a + b}$. **10.** 1. **11.** b. **12.** $\frac{a^2bc^2 + ab + c^2}{a^2 - bc}$. **13.** 1. **14.** $\frac{ab}{a + b}$.
15. $\frac{8a}{25}$. **16.** -4. **17.** 6. **18.** $-7a/3$. **19.** $8\frac{7}{8}$. **20.** 78, 24.
21. 5, 26, 27. **22.** 51 mi. per hr., 34 mi. per hr. **23.** 16 days.
24. \$2100. **25.** \$12,000. **26.** 4 years. **27.** 2 hrs. **28.** 720 miles.
29. A, 11 das.; B, 22 das.; C, 33 das. **30.** $12\frac{3}{4}\frac{6}{7}$ mins.

Pages 29–32

1. 7, 10. **2.** $-\frac{15}{22}, \frac{1}{22}$. **3.** $\frac{1}{4}, \frac{1}{2}$. **4.** $a + b, a - b$. **5.** $\frac{1}{3}, \frac{1}{5}$.
6. $\frac{7}{4}$, 2. **7.** 7, 3. **8.** $2b, 3a$. **9.** $\frac{2a}{a - b}, \frac{2b}{a + b}$. **10.** -3, 4.
11. Equations not independent. **12.** 5, 2. **13.** 2, 3, 1. **14.** $c, c, ab/c$.
15. Equations not consistent. **16.** $a^2 + 2bc, b^2 + 2ca, c^2 + 2ab$.
17. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. **18.** 6, 25. **19.** 5 miles per hr., 15 miles per hr. **20.** $\frac{5}{9}$.
21. 3 dimes, 4 quarters, 5 half-dollars. **22.** \$25,000; \$15,000. **23.** 6, 8.
24. 84, 48. **25.** 357. **26.** 851. **27.** 4 hrs., 6 hrs. **28.** 48 days, $28\frac{4}{5}$ days, 144 days, 16 days. **29.** 42 yrs., 7 yrs., 11 yrs. **30.** 300 horses, 60 days.

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1. 3, 4. **2.** 2, $-\frac{5}{6}$. **3.** 2, 2, 3. **4.** Equations not independent.
5. 5, 4, 3. **6.** 1, 2, 3. **7.** 0, 0, 0. **8.** 3, 5, -1. **9.** $\frac{18}{5}, \frac{17}{5}$, 1.
10. $-abc, ab + bc + ca, -(a + b + c)$.

Pages 38–39

1. $a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$.
2. $a^7 - 7a^6b + 21a^5b^2 - 35a^4b^3 + 35a^3b^4 - 21a^2b^5 + 7ab^6 - b^7$.

3. $32a^5 + 80a^4x + 80a^3x^2 + 40a^2x^3 + 10ax^4 + x^5$.

4. $a^5 - 10a^4b + 40a^3b^2 - 80a^2b^3 + 80ab^4 - 32b^5$.

5. $p^5 + 10p^4q + 40p^3q^2 + 80p^2q^3 + 80pq^4 + 32q^5$.

6. $c^4 - 20c^3d + 150c^2d^2 - 500cd^3 + 625d^4$.

7. $x^6 + 18x^5y^2 + 135x^4y^4 + 540x^3y^6 + 1215x^2y^8 + 1458xy^{10} + 729y^{12}$.

8. $32 + 240x^2 + 720x^4 + 1080x^6 + 810x^8 + 243x^{10}$.

9. $x^{18} - 6x^{15}y^2 + 15x^{12}y^4 - 20x^9y^6 + 15x^6y^8 - 6x^3y^{10} + y^{12}$.

10. $x^7 - 7x^6y^2 + 21x^5y^4 - 35x^4y^6 + 35x^3y^8 - 21x^2y^{10} + 7xy^{12} - y^{14}$.

11. $x^8y^8 - 8x^7y^7 + 28x^6y^6 - 56x^5y^5 + 70x^4y^4 - 56x^3y^3 + 28x^2y^2 - 8xy + 1$.

12. $729m^{12} + 2916m^{10}n + 4860m^8n^2 + 4320m^6n^3 + 2160m^4n^4 + 576m^2n^5 + 64n^6$.

13. $a^8 - \frac{12a^6}{b} + \frac{54a^4}{b^2} - \frac{108a^2}{b^3} + \frac{81}{b^4}$.

14. $64a^3 + 24a^2b + 3ab^2 + \frac{b^3}{8}$.

15. $\frac{64}{a^6} - \frac{192}{5a^3} + \frac{48}{5} - \frac{32a^3}{25} + \frac{12a^6}{125} - \frac{12a^9}{3125} + \frac{a^{12}}{15625}$.

16. $\frac{32a^{10}}{b^5} + \frac{80a^7}{3b^2} + \frac{80a^4b}{9} + \frac{40ab^4}{27} + \frac{10b^7}{81a^2} + \frac{b^{10}}{243a^5}$.

17. $256a^4 - \frac{256a^3b}{3} + \frac{96a^2b^2}{9} - \frac{16ab^3}{27} + \frac{b^4}{81}$.

18. $\frac{x^6}{a^6} - \frac{6x^4}{a^4} + \frac{15x^2}{a^2} - 20 + \frac{15a^2}{x^2} - \frac{6a^4}{x^4} + \frac{a^6}{x^6}$.

19. $64x^6 - 96x^5a + 60x^4a^2 - 20x^3a^3 + \frac{15x^2a^4}{4} - \frac{3xa^5}{8} + \frac{a^6}{64}$.

20. $216a^3 - 54a^2b + \frac{9ab^2}{2} - \frac{b^3}{8}$.

21. $\frac{625a^4}{16} - \frac{250a^3b}{3} + \frac{200a^2b^2}{3} - \frac{640ab^3}{27} + \frac{256b^4}{81}$.

22. $\frac{x^{10}}{32y^5} + \frac{5x^7}{8y^2} + 5x^4y + 20xy^4 + \frac{40y^7}{x^2} + \frac{32y^{10}}{x^5}$.

23. $\frac{81x^{12}}{16a^8} - \frac{54x^7}{a^5} + \frac{216x^2}{a^2} - \frac{384a}{x^3} + \frac{256a^4}{x^8}$.

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1. $-35a^4x^3$. **2.** $144a^2x^7$. **3.** $\frac{55a^8b^4}{144}$. **4.** $\frac{135a^8x^4}{256}$. **5.** $-13{,}608a^3x^{10}$.

6. $5280m^{21}n^8$. **7.** $\frac{210a^4}{x^4}$. **8.** $-70p^{26}q^3$. **9.** $\frac{896a^{10}b^{10}}{27}$.

10. $\frac{35c^{26}d^{26}x^2}{3}$. **11.** $6864a^{14}b^{12}$. **12.** $-3432m^7n^{14}$.

Pages 42–43

17. $\frac{4a^2y^{9/2}z^3}{5x^{2/3}}$. 18. $\frac{6b^{5/2}c^2}{7a^2}$. 19. $\frac{a^2x^{11}}{3y^6}$. 20. $\frac{4bc}{5a^2}$. 21. $\frac{3b^4}{8a^2n^3}$.

22. $\frac{5y^6z^3}{8x^3}$. 23. $\frac{4x^{7/12}}{7y^{1/3}}$. 24. $\frac{9a^2b^2}{2n^5}$. 25. $\frac{27b}{8a}$. 26. 32.

27. ± 27. 28. $\pm\frac{32}{243}$. 29. $\pm\frac{1}{8}$. 30. $\pm\frac{1}{32}$. 31. $\frac{1}{16}$. 32. ± 2.

33. ± 1. 34. 36. 35. 1. 36. $-\frac{1}{8}$. 37. $\pm\frac{1}{128}$.

38. $a^{5/2} - a^{3/2}$. 39. $x^2 - y^3$. 40. $6x^2 - 5x - 5 + 5x^{-1} - x^{-2}$.
41. $2x^{3/4} - 7 - 17x^{-3/4} + 19x^{-3/2} + 21x^{-9/4} - 18x^{-3}$.
42. $9x^{4/5} - 9x^{2/5} - 25 + 23x^{-2/5} + 6x^{-4/5}$. 43. $a + b$. 44. $x - y$.
45. $x^{2/3} + x^{1/3} + 1$. 46. $x^{3/2} - x + x^{1/2} - 1$. 47. $a^{2/3} - a^{-2/3}$.
48. $x^{-3/4} + 3x^{1/2} + 3x^{7/4} + x^3$. 49. $3a^{1/3} - 5a^{-1/3} + 2a^{-1}$.
50. $x^{1/2}y^{-5/3} - 2y^{-4/3} - 3x^{-1/2}y^{-1}$. 51. $a^{4/3} - 2a + 3a^{2/3} - 2a^{1/3} + 1$.
52. $2a^{3/4}b^{-2} + a^{1/4} - 4a^{-1/4}b^2$. 53. $b^{2n}x^{-2m} - 2a^nb^n + a^{2n}x^{2m}$.
54. $16a^2 - 8 - 15a^{-2}$. 55. $a^2 + 2a^{3/2} + a - 16$.
56. $8a^4 - 36a^{8/3}b^{-1/2} + 54a^{4/3}b^{-1} - 27b^{-3/2}$.
57. $x^3 - 6x^{5/2}y^{-2/3} + 15x^2y^{-4/3} - 20x^{3/2}y^{-2} + 15xy^{-8/3} - 6x^{1/2}y^{-10/3} + y^{-4}$.
58. $a^{-1} + 4a^{-3/4}b^{1/4} + 6a^{-1/2}b^{1/2} + 4a^{-1/4}b^{3/4} + b$. 59. $\frac{35a^{-1}y^{10/3}}{8x^{8/3}}$. 60. $-\frac{15x^9}{4a^{21}}$.
61. $81a^2x^{-1}y^{2/3}$. 62. $256a^{-2}x^{-1}$. 63. $a^2b^{-1}c^{-2}xy^2z^{-3}$. 64. $a^{5/3}$.
65. $\frac{a^{1/5}x^{6/5}}{4}$. 66. $\frac{a^2}{b^3}$. 67. $\frac{x^7}{y^7}$. 68. ± 125. 69. $\frac{1}{1024}$. 70. $\pm\frac{1}{64}$.

Pages 45–46

1. $2\sqrt{3}$. 2. $3\sqrt[4]{2}$. 3. $12\sqrt{2}$. 4. $3\sqrt[4]{2}$. 5. $\sqrt[3]{5}$. 6. $3ab\sqrt[5]{ab^2}$.
7. $2(a+b)\sqrt{a+b}$. 8. $4(x-y)\sqrt[3]{(x-y)^2}$. 9. $ab^2\sqrt[3]{5a^2}$.
10. $10ax^3\sqrt{5abx}$. 11. $\frac{\sqrt{10}}{20}$. 12. $\frac{3a\sqrt{b}}{4xy^3}$. 13. $\frac{4a^4\sqrt{5bx}}{15b^3x^2}$.
14. $\frac{2xy\sqrt[6]{18a^2bx^2}}{3ab}$. 15. $7\sqrt{2}$. 16. $13\sqrt{2}$. 17. $3\sqrt[3]{3a}$.
18. $(a+3)\sqrt{a-1}$. 19. $\frac{5\sqrt{2}}{6} + \frac{7\sqrt{30}}{18}$. 20. $43\sqrt[3]{2}$. 21. $7\sqrt[3]{2}$.
22. $20\sqrt{5} - 11\sqrt{3}$. 23. $2a\sqrt{a-b}$. 24. $6\sqrt{2}$. 25. $216\sqrt{10}$.
26. 96. 27. 3. 28. $-30 - \sqrt{18}$. 29. $-14 + 2\sqrt{10}$. 30. $2\sqrt{6}$.
31. $-30 - 5\sqrt{6} + \sqrt{15} + 11\sqrt{10}$. 32. $2\sqrt{5}$. 33. $2 + 5\sqrt{6}$. 34. 2.

35. $5\sqrt{2} - \frac{1}{2}$. **36.** $26 - 4\sqrt{30}$. **37.** $252\sqrt{3} - 149\sqrt{5}$.

38. $97 - 56\sqrt{3}$. **39.** $2884 - 912\sqrt{10}$.

40. $a^5 + 5a^4\sqrt{bc} + 10a^3bc + 10a^2bc\sqrt{bc} + 5ab^2c^2 + b^2c^2\sqrt{bc}$.

41. $(a + 13)\sqrt{a + 1} + 6a + 14$. **42.** $\frac{19 - 8\sqrt{3}}{13}$. **43.** $-\frac{7 + 2\sqrt{6}}{5}$.

44. $\frac{8 - 5\sqrt{2}}{7}$. **45.** $\frac{a^2 + b + 2a\sqrt{b}}{a^2 - b}$. **46.** $\frac{a + 5 - 4\sqrt{a + 1}}{a - 3}$.

47. $\frac{x - \sqrt{x^2 - y^2}}{y}$. **48.** 1.984 **49.** 0.318 **50.** 3.943 **51.** 0.69

Page 49

1. -1. **2.** $-i$. **3.** i. **4.** $7i$. **5.** $18\sqrt{-2}$. **6.** $60i$. **7.** $10\sqrt{-3}$.

8. -6. **9.** -18. **10.** $21 + 7i\sqrt{3}$. **11.** 29. **12.** 154. **13.** $\sqrt{-2}$.

14. $-3i$. **15.** 2. **16.** $7\sqrt{3}$. **17.** $-729i$. **18.** $\frac{8 + i}{5}$. **19.** $-7 - 24i$.

20. $-\frac{i}{2}$. **21.** $-3\sqrt{3} - 7\sqrt{-2}$. **22.** $\frac{a^2 - b^2 + 2abi}{a^2 + b^2}$.

CHAPTER II

Page 52

1. $5, -1$. **2.** $a, -a$. **3.** $15, 0$. **4.** $-1, -\frac{1}{6}$. **5.** $-4, \frac{9}{2}$. **6.** $2, -\frac{7}{6}$.

7. $\frac{2}{3}, -\frac{3}{2}$. **8.** $2, -\frac{4}{3}$. **9.** $-4, \frac{8}{7}$. **10.** $2a, \frac{a}{2}$. **11.** $9, -5$.

12. $3, -3$. **13.** $0, 3, -3$. **14.** $0, a, -a$. **15.** $14, -14$. **16.** $a, -a$.
17. $2, -1, -2$. **18.** $1, 2, -3$. **19.** $0, 0, 9, -9$. **20.** $1, 4, -1, -4$.

21. $1, 2, 3$. **22.** $a, -\frac{1}{a}$. **23.** $5, -5$. **24.** $a - b, -a - 3b$.

Pages 56–57

1. $9, -6$. **2.** $-10, -16$. **3.** $\frac{-5 \pm 5\sqrt{5}}{2}$. **4.** $10, -\frac{28}{3}$.

5. $\frac{7 \pm \sqrt{241}}{6}$. **6.** $-2, \frac{5}{7}$. **7.** $19, -4$. **8.** $2, -\frac{7}{2}$. **9.** $-2 \pm 3\sqrt{6}$.

10. $\frac{1 \pm i\sqrt{149}}{5}$. **11.** $3, -\frac{11}{2}$. **12.** $-10 \pm \sqrt{97}$. **13.** $3, -\frac{4}{5}$.

14. $1, -16$. **15.** $\frac{a \pm 3ai\sqrt{3}}{2}$. **16.** $12, -2$. **17.** $\frac{3}{4}, \frac{1}{8}$. **18.** $\frac{1 \pm \sqrt{19}}{3}$.

19. $3, -\frac{6}{5}$. **20.** $(a+b)^2, -(a-b)^2$. **21.** $\frac{-4m \pm i\sqrt{39m^2+55}}{1+m^2}$.

22. $5\sqrt{ab}, -3\sqrt{ab}$. **23.** $a+b, a-b$. **24.** $2a-3, 3$. **25.** $\frac{4}{a}, -\frac{2}{a}$.

26. $-1, 0$. **27.** $3, -5$. **28.** $\frac{a^2+b^2}{2a}, \frac{a^2+b^2}{2b}$. **29.** $5, -\frac{5}{4}$.

30. $-2, \frac{a-b+c}{a-b}$. **31.** $5, \frac{23}{7}$. **32.** $4, \frac{3}{2}$.

Pages 58–59

1. $2 \pm i\sqrt{7}$. **2.** $4 \pm i\sqrt{19}$. **3.** $\frac{7 \pm i\sqrt{31}}{8}$. **4.** $-2, \frac{5}{7}$. **5.** $\frac{1}{3}, -\frac{2}{3}$.

6. $\frac{2}{3}, -\frac{5}{2}$. **7.** $\frac{3 \pm i\sqrt{23}}{8}$. **8.** $2, 11$. **9.** $4, \frac{11}{2}$. **10.** $a, -5a$.

11. $1, -2$. **12.** $\frac{-1 \pm \sqrt{5}}{2}$. **13.** $\frac{3}{a}, -\frac{7}{a}$. **14.** $a \pm bi$. **15.** $2, \frac{5}{a}$.

16. $1, \frac{b}{a-b}$. **17.** $m+n, -2m$. **18.** $5, -\frac{5}{2}$. **19.** $-a, b$.

20. $\frac{ab \pm a\sqrt{b^2-1}}{2}$. **21.** $a+b, \frac{2ab}{a+b}$. **22.** $8, -\frac{3}{2}$. **23.** $2, \frac{37}{2}$.

24. $-a, -b$. **25.** $\frac{a+1}{2}, \frac{2}{a+1}$. **26.** $\frac{ab+1}{a}, \frac{ab+1}{b}$. **27.** $1, \frac{a}{a+b+c}$.

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4. $6x^2-x-2=0$. **5.** $4x^2+5x-6=0$. **6.** $mnx^2-(m^2+n^2)x+mn=0$.
7. $x^2-4x+1=0$. **8.** $x^2+1=0$. **9.** $x^2-2ax+a^2-b^2=0$.
10. $x^2-2ax+a^2-b=0$. **11.** $x^2-6x+10=0$. **12.** $4x^2-4\sqrt{a}x+a-9b=0$.
13. $x^2+x+1=0$. **14.** $x^2-3x+10=0$.

Pages 63–64

12. $-\frac{8}{3}$. **13.** -3. **14.** ± 8. **15.** $\frac{9}{2}$. **16.** 4. **17.** $5, -7$.

18. $3, -5$. **19.** $-\frac{77}{8}$. **20.** $-5 \pm 2\sqrt{10}$. **21.** $1, -\frac{1}{2}$.

22. $\pm \sqrt{a^2m^2+b^2}$. **23.** $5, -\frac{7}{25}$. **24.** $-\frac{b}{c}$. **25.** $\frac{b^2-2ac}{ac}$.

26. $\frac{b^2 - 2ac}{a^2}$. **27.** $\frac{b^2 - 4ac}{a^2}$. **28.** $\frac{b^2 - ac}{a^2}$. **29.** $\frac{b^2 - 2ac}{c^2}$.

30. $ax^2 - bx + c = 0$. **31.** $ax^2 + 2bx + 4c = 0$.

32. $ax^2 + 3bx + 9c = 0$. **33.** $bx^2 + bx + a = 0$.

34. $a^2x^2 - (b^2 - 2ac)x + c^2 = 0$. **35.** $c^2x^2 - (b^2 - 2ac)x + a^2 = 0$.

Pages 66–67

1. $\pm 3, \pm 4$. **2.** $\pm 3, \pm \frac{\sqrt{6}}{3}$. **3.** $-\frac{1}{2}, \sqrt[5]{-5}$. **4.** 16.

5. 625. **6.** $\pm\frac{1}{2}, \pm\frac{i\sqrt{10}}{4}$. **7.** 1. **8.** $1, \frac{5\sqrt[3]{50}}{4}$. **9.** $-1, -\frac{1}{32}$.

10. $\frac{1}{4}, -\frac{1}{2}$. **11.** $2, -\frac{1}{2}, \frac{1 \pm \sqrt{10}}{3}$. **12.** 256. **13.** $\frac{1}{2}, -\frac{1}{3}$.

14. 1. **15.** $1, 2, \frac{3 \pm \sqrt{5}}{2}$. **16.** ± 8. **17.** $\pm 2, \pm i\sqrt{5}$.

18. $\pm 3, \pm\sqrt{10}$. **19.** 9. **20.** $2, -8$.

21. $\frac{-5 \pm \sqrt{13}}{2}, \frac{3 \pm i\sqrt{3}}{2}$. **22.** $1, \frac{7}{4}, \frac{13 \pm \sqrt{161}}{4}$.

Pages 68–69

1. $\frac{3}{2}, -\frac{1}{2}$. **2.** 51. **3.** $\frac{1}{4}$. **4.** 2, 3. **5.** 2, 3. **6.** 2. **7.** $\frac{5}{4}, \frac{13}{12}$.

8. 8. **9.** 4, 8. **10.** -2. **11.** 5. **12.** 3, -1. **13.** a, b. **14.** 6, -3.

15. $\frac{1}{24}$. **16.** -1. **17.** $2, \frac{2}{3}$. **18.** 3, -2.

Pages 70–72

1. $\frac{4}{5}, \frac{5}{4}$. **2.** 15, 16; $-16, -15$. **3.** 12, 14; $-14, -12$. **4.** $12\frac{1}{2}$ ft., 16 ft. **5.** 9, 12, 15. **6.** 22, 23; $-23, -22$. **7.** 11 ft., 16 ft.

8. 30 miles per hr. **9.** 5 men, 9 women. **10.** $4\frac{1}{2}$ seconds. **11.** $7\frac{1}{2}$ seconds.

12. $2\frac{1}{2}$ seconds. **13.** 3 inches. **14.** 9, 21; $-\frac{3}{2}, \frac{63}{2}$. **15.** 12 inches.

16. 50 in., 120 in. **17.** \$65, \$35. **18.** 2, 3, 4, 5; $-5, -4, -3, -2$.

19. 40 mi. per hr., 34 mi. per hr. **20.** $\frac{5}{3}, -\frac{3}{5}$. **21.** 6 hrs.

22. 12 das., 24 das. **23.** $1\frac{1}{2}$ hrs., 4 mi. per hr. **24.** 68.

CHAPTER III

Pages 76–77

1. 16 : 25. **2.** 9 : 14. **3.** 25 : 14. **4.** 115 : 102. **5.** 3 : 10. **6.** 1 : 5.
7. $\sqrt{2}$: 1. **8.** 1 : 2. **9.** 5/8. **10.** ± 4.4 **11.** $\pm \frac{a+b}{ab}$. **12.** 8.
13. 28, 42. **14.** 2. **15.** 8, 13. **16.** 3, 4, 27, 36 or 3, 27, 4, 36.
17. 300 rev. per min. **18.** 15 : 7. **19.** 55 : 1, approx. **33.** $-b/a$.
34. 0, 2.

CHAPTER IV

Pages 80–81

1. 39, 400. **2.** $-26, -114$. **3.** 51, 345. **4.** $-\frac{131}{10}, -\frac{335}{2}$.
5. $87x$, $801x$. **6.** $12a + 7b$, $75a + 25b$. **7.** $\frac{12a-23b}{a+b}, \frac{78a-144b}{a+b}$.
8. $d = 5$, $s = -363$. **9.** $a = 4$, $s = 175$. **10.** $n = 11$, $l = -\frac{11}{6}$;
$n = 21$, $l = \frac{3}{2}$. **11.** $n = 7$, $d = 2x$. **12.** $\frac{l-a}{n-1}$.
13. $a = \frac{2s-nl}{n}$, $d = \frac{2(nl-s)}{n(n-1)}$. **14.** 800 ft. **15.** \$2754.

Pages 82–84

1. -264. **2.** 600. **3.** $-4, 1, 6, 11, 16, 21, 26, 31$.
4. $\frac{2}{3}, \frac{7}{12}, \frac{1}{2}, \frac{5}{12}, \frac{1}{3}, \frac{1}{4}$. **5.** 5. **6.** 41. **7.** 3. **8.** $-\frac{13}{3}$.
9. $-22, -19, -16, -13, \cdots$. **10.** 131. **11.** 12,500. **12.** 9.
13. 1,610 ft. **14.** 5.9 secs. **15.** 180. **16.** 6000. **17.** 13,860. **18.** 7.
19. 135. **20.** \$344.50. **21.** $-6, -2, 2, 6, 10$. **22.** 2 : 1. **23.** \$4262.50.
27. 2680 ft. **28.** 246. **29.** \$12.75. **30.** \$50.50. **31.** 4 hrs., 9 hrs.

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1. 1536, 3069. **2.** 256, 172. **3.** 486, 364. **4.** 256, 511. **5.** $\frac{1}{2048}, \frac{2047}{2048}$.
6. 6144, $\frac{39321}{8}$. **7.** $rl + (1-r)s$. **8.** 4, 512. **9.** $\frac{384}{125}$, 4. **10.** 2, 5.
11. 5, 77,777. **12.** 8th.

Pages 91–92

1. $\pm\frac{5}{2}$. **2.** $\pm(2x^2 - 5xy - 3y^2)$. **3.** $\frac{16}{3}$, 8, 12, 18, 27, or $-\frac{16}{3}$, 8, -12, 18, -27. **4.** 6, 3, $\frac{3}{2}$, $\frac{3}{4}$, $\frac{3}{8}$, $\frac{3}{16}$. **5.** 27, 9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, or -27, 9, -3, 1, $-\frac{1}{3}$, $\frac{1}{9}$, $-\frac{1}{27}$. **6.** 27. **7.** $\frac{12}{5}$. **8.** 18. **9.** $-\frac{12}{11}$. **10.** $\frac{24}{25}$. **11.** $\frac{8}{15}$. **12.** $-\frac{21}{8}$. **13.** $\frac{162}{7}$. **14.** $\frac{64}{7}$. **15.** $-\frac{3}{275}$. **16.** $\frac{7}{3}$. **17.** 5. **18.** $\frac{3}{11}$. **19.** $\frac{47}{99}$. **20.** $\frac{763}{3300}$. **21.** $\frac{341}{999}$. **22.** $\frac{1066}{4995}$. **23.** $\frac{299}{550}$. **24.** $\frac{7}{90}$. **25.** 3, 12, 48, 192.

26. 2, 6, 10, 14, 18, $\cdots$, or 2, 2, 2, 2, 2, $\cdots$. **27.** 4.

28. 1, 3, 9, 27, 81, 243, or $-\frac{14}{13}$, $\frac{42}{13}$, $-\frac{126}{13}$, $\frac{378}{13}$, $-\frac{1134}{13}$, $\frac{3402}{13}$.

29. 3, 6, 12. **30.** 1, 2, 4, 8. **31.** 8 ins. **32.** 160 ft. **33.** 248. **35.** \$800. **39.** \$655.35.

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1. $\frac{6}{31}$. **2.** $-\frac{12}{25}$. **3.** $\frac{1}{53}$. **4.** $-\frac{4}{75}$. **5.** $\frac{1}{18}$. **6.** $\frac{1}{a}$. **7.** $\frac{18}{5}$, $\frac{9}{2}$, 6, 9. **8.** $-\frac{2}{3}$, -2, 2, $\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$. **9.** 7, 5. **10.** -14, -42.

CHAPTER V

Page 100

5. 20. **6.** 12. **7.** $\frac{4}{3}$. **8.** -10. **9.** -6. **10.** 1. **11.** 0.7781 **12.** 2.0791 **13.** 1.5741 **14.** -1.6990 **15.** 0.1267 **16.** 0.1544

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3. $8.3856 - 10$. **6.** $6.3856 - 10$.

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1. 1.3802 **2.** 1.5106 **3.** 0.6800 **4.** $9.3310 - 10$. **5.** 2.9721 **6.** 2.7345 **7.** 1.4570 **8.** 0.9204 **9.** $8.6571 - 10$. **10.** 0.0102 **11.** 1.2165 **12.** $6.6021 - 10$. **13.** 1.8008 **14.** $7.2625 - 10$. **15.** 3.5386 **16.** 0.7189

Page 109

1. 25.9 **2.** 136.3 **3.** 5.661 **4.** 90. **5.** 172.1 **6.** 1.568 **7.** 99.65 **8.** 5664. **9.** 140.5 **10.** 37,940. **11.** .1768 **12.** .03422 **13.** .0002214 **14.** .000000564 **15.** .0000971

Pages 112–113

1. 1008. **2.** 57,930. **3.** 219,000. **4.** 0.03427 **5.** 24880. **6.** 16.33 **7.** 4. **8.** 9.462 **9.** 163.3 **10.** 819. **11.** .00001198 **12.** 108,300,000. **13.** .000001522 **14.** 162.4 **15.** 5.514 **16.** -2.202 **17.** 6.343 **18.** 3.403 **19.** 0.4352 **20.** 1000. **21.** 13.85 **22.** 2.266 **23.** 6.147 **24.** 0.6194 **25.** 29.24 **26.** 0.2332 **27.** 0.202 **28.** 1.234 **29.** -122.1 **30.** 2.512 **31.** -2.122 **32.** 1.055 **33.** Seven. **34.** 22,180 cu. in. **35.** 40.2 cu. in. **36.** 6,201 sq. ft. **37.** 170,700 lbs.

Page 113

1. 1.5 **2.** 0.5 **3.** 1.43 **4.** -0.48 **5.** 0.96 **6.** -0.17 **7.** 3.2 **8.** 0.84 **9.** 3 or -1. **10.** $\dfrac{\log [a - s(1-r)] - \log a}{\log r}$.

CHAPTER VI

Page 121

1. 3.16 **2.** 2.96 **3.** 11.09 **4.** 1.10 **5.** 6.03 **6.** 2.97 **7.** 8.98 **8.** 2.01

9. $1 + \dfrac{1}{3}x - \dfrac{1}{9}x^2 + \dfrac{5}{81}x^3 + \cdots$.

10. $a^{1/5} - \dfrac{1}{5}a^{-4/5}x - \dfrac{2}{25}a^{-9/5}x^2 - \dfrac{6}{125}a^{-14/5}x^3 + \cdots$.

11. $x^{-4} + 4x^{-5}y^2 + 10x^{-6}y^4 + 20x^{-7}y^6 + \cdots$.

12. $1 - 8x + 40x^2 - 160x^3 + \cdots$.

13. $\dfrac{1}{64a^6} + \dfrac{15b}{64a^7} + \dfrac{525b^2}{256a^8} + \dfrac{875b^3}{64a^9} + \cdots$.

14. $\dfrac{b^5}{a^5} + \dfrac{5b^7}{a^7} + \dfrac{15b^9}{a^9} + \dfrac{35b^{11}}{a^{11}} + \cdots$. **15.** $-\dfrac{4}{81}a^{-7/3}x^3$. **16.** $\dfrac{45y^2}{32x^6}$.

17. x^6. **18.** $-21a^{10}b^{-5/3}$. **19.** $\dfrac{4389}{8}a^{-115/8}b^{-10}$. **20.** $35c^{16}e^{-2}$.

CHAPTER VIII

Pages 134–135

2. $c = \pi d$. **4.** $A = \frac{1}{2}bh$. **6.** $v = \frac{\pi}{6}d^3$. **7.** 10 days. **8.** 27 : 64. **9.** $\frac{64}{7}$.
10. $y = 3x^2z^3$. **12.** $8y = 5x^3$. **15.** 72. **16.** 33⅓%. **17.** 6. **18.** 258 ft.
19. 152 : 100. **20.** 20.

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1. $-2, 0, -8, 2, -4$. **2.** $-7, 0, -15, 33, 0$. **3.** $A = \pi d^2/4$.
4. $y = \sqrt{100 - x^2}$. **5.** $\pm 4, \pm 2\sqrt{3}, \pm \sqrt{7}, 0$.

CHAPTER IX

Page 149

1. 7. **2.** -4. **3.** -2. **4.** 5. **5.** 21. **6.** -1.

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1. 2. **2.** m. **3.** $2x - 3$. **4.** $-2x + 2$. **5.** $3x^2 - 8x + 6$.
6. $3x^2 + 8x - 5$. **7.** $6x^5 + 12x^3 - 2$. **8.** $6x - 4$. **9.** $2mx^{2m-1} + 4mx^{m-1}$.
10. $40x^7 - 15x^4 + 4x$. **11.** $0, -2, 2$. **12.** 9. **15.** (1, 4). **16.** 3.
18. (2, 4).

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1. Minimum, -7. **2.** Minimum, -18. **3.** Maximum, 11.
4. Minimum, 3. **5.** Minimum, 0. **6.** Maximum, 0. **7.** Minimum, -16.
8. Maximum, 16. **12.** The square whose side is 10 in.
13. 50 rods × 50 rods. **14.** No. **15.** 156 ft. 3 in.

CHAPTER X

Pages 170–171

1. $-2, -3; -\frac{1}{17}, -\frac{29}{17}$. **2.** 14, 2; 2, 14. **3.** $0, -2; 4, 10$.
4. $7, -5; 5, -7$. **5.** $\frac{1}{15}, \frac{2}{15}$. **6.** $1, 3; -1, -3; 39, -\frac{29}{3}; -39, \frac{29}{3}$.
7. $2, 3; -2, -3; 5, 6; -5, -6$.
8. $3, 4; -\frac{33}{8}, -\frac{11}{2}; \frac{-39 \pm 3i\sqrt{183}}{16}, \frac{52 \mp 4i\sqrt{183}}{16}$.

9. $1, 2; -1, -2; \frac{3\sqrt{21}}{7}, \frac{2\sqrt{21}}{7}; \frac{-3\sqrt{21}}{7}, \frac{-2\sqrt{21}}{7}$.

10. $0, 2\sqrt{11}; 0, -2\sqrt{11}; 4, 2; -4, -2$.
11. $1, 3; -1, -3; 2, 1; -2, -1$.

12. $12, 3; -7, -\frac{7}{4}; \frac{3 \pm 3\sqrt{29}}{2}, \frac{1 \pm \sqrt{29}}{2}$.

13. $4, 3; -4, -3; \frac{9}{2}, \frac{8}{3}; -\frac{9}{2}, -\frac{8}{3}$.

14. $6, -20; -6, 20; 2, 4; -2, -4$. **15.** $6, 3; -6, -3; 3, 1; -3, -1$.
16. $3, 2; 2, 3; -6, -1; -1, -6$. **17.** $8, 9; -8, -9$.
18. $6, 3; -6, -3$. **19.** 12 ft., 16 ft. **20.** $49, 17; -17, -49$.
21. 600 ft., 800 ft. **22.** $27, 8; -8, -27$. **23.** 75. **24.** 12 ft., 16 ft.

25. 8 ft., 10 ft. **26.** 12 in., 14 in. **27.** $\frac{72}{5}$. **28.** $12, 15; -5, -2$.

29. 73, 37. **30.** $7, 9; -7, -9$. **31.** $15, 11; -1, -5$. **32.** 24.
33. A, 6 hrs.; B, 4 hrs.

CHAPTER XI

Pages 187–189

1. $1, 1, 6$. **2.** $1, 2, -3$. **3.** $1, 1, 2, -2$. **4.** $2, 4, -1 \pm i\sqrt{3}$.

5. $2, -2, 3 \pm \sqrt{5}$. **6.** $-2, -4, 3 \pm \sqrt{3}$. **7.** $3, 4, \frac{1}{2}(-1 \pm i\sqrt{3})$.

8. $2, 4, -3, -4$. **9.** $x^3-6x^2+36x-162=0$. **10.** $x^3+x^2-8x+12=0$.
11. $x^3 - 2x^2 + 2x - 3 = 0$. **12.** $x^4 - 9x^2 + 81x + 567 = 0$.
13. $x^5 + 6x^3 - 8x + 8 = 0$. **14.** $x^4 - 5x^3 - 2x^2 - 8x - 24 = 0$.
15. $x^3 - 3x^2 - 2x + 16 = 0$. **16.** $x^4 - 4x^3 + 108x - 1728 = 0$.
17. $x^3-6x-9=0$. **18.** $x^4-x^3+3x^2+4=0$. **19.** $5x^3-2x^2-x-3=0$.
20. $2x^3 - 3x^2 - 6 = 0$. **21.** $x^4 + 3x^3 + 4x^2 + 7x + 6 = 0$.

22. $3x^4+5x^3-2x^2+3x-6=0$. **23.** $\frac{1}{2}, \frac{1}{2}(-5 \pm \sqrt{37})$. **24.** $-\frac{1}{2}, -1, 4$.

25. $\frac{5}{4}, \pm\frac{i\sqrt{6}}{2}$. **26.** $\frac{1}{2}, \frac{3}{2}, \pm i\sqrt{5}$. **27.** $2, -2, -\frac{2}{3}, \frac{3}{2}$.

28. $1, -2, \frac{1}{2}, \frac{2}{3}$. **29.** $4, 5, \pm \sqrt{3}$. **30.** $-\frac{1}{2}, -2, \frac{1}{2}(-1 \pm i\sqrt{3})$.

31. $1, 3, -2, -\frac{1}{3}$. **32.** $-2, -2, \frac{5}{3}, \frac{5}{3}$. **33.** $2, -3, 4, \pm i$.

34. $3, -6$. **35.** $2, 2, -3, 1 \pm i$. **36.** $2, 5, \pm\frac{1}{2}\sqrt{-2 \pm 2i\sqrt{15}}$.

37. $1, -1, -2, -\frac{3}{2}, -1 \pm i\sqrt{2}$. **38.** $-1, \pm 1, \pm 2, \pm 3$.

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2. No negative roots. **3.** No negative roots; at least 2 imaginary roots. **5.** At least 2 imaginary roots. **7.** At least 2 imaginary roots.

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1. 1, 2. **2.** $-2, -1$. **3.** $-2, -1$; 0, 1; 7, 8. **4.** $-2, -1$. **5.** -1, 0; 0, 1; 1, 2. **6.** -1, 0; 0, 1; 1, 2. **7.** 0, 1; 3, 4. **8.** 2 roots between -1 and 0; 2 roots between 2 and 3. **9.** $-3, -2$; $-2, -1$; 0, 1; 3, 4. **10.** -1, 0; 2, 3. **11.** 0.2 **12.** -0.3 **13.** -1.2 **14.** 2.4 **15.** 2.1 **16.** 0.6

Page 197

1. 2.095 **2.** 0.4408 **3.** -2.892 **4.** 1.214 **5.** 3.874 **6.** 2.816

Pages 205–206

1. 3.154 **2.** 2.67 **3.** -2.838 **4.** 2.769 **5.** -3.531 **6.** 1.165 **7.** -1.396 **8.** 1.202 **9.** 3.693 **10.** 2.427; 2.757 **11.** -0.142; -0.63

CHAPTER XII

Page 209

1. 12. **2.** 60. **3.** 60. **4.** 1000. **5.** 14,400. **6.** 35,152; 1352. **7.** 625. **8.** 48. **9.** 120,960; 5040. **10.** 325.

Pages 212–213

1. 3024. **2.** 720. **3.** 2,162,160. **4.** 657,720. **5.** 9. **6.** 3. **7.** 7. **9.** 5040. **10.** 360. **11.** 60,480. **12.** 64; 340. **13.** 3360. **14.** 720; 120. **15.** 3360; 14,968,800; 64,864,800; 277,200. **16.** 630,630. **17.** 216. **18.** 420. **19.** 3600. **20.** 120. **21.** 2880. **22.** 100,800. **23.** 720; 1440. **24.** 18. **25.** 50,400. **26.** 3,628,800. **27.** 30,240.

Pages 215–217

1. 210. **2.** 220. **3.** 5005. **4.** 12. **7.** 6. **8.** 45. **9.** 54. **10.** 28. **11.** 13,860. **12.** 22,680. **13.** 8106. **14.** 43,092,000. **15.** $103{,}950 \times 14!$ **16.** 4845; 969. **17.** 10; 35; 185. **18.** 21,600. **19.** 210; 371. **22.** 63. **23.** 315. **24.** 1050; 1414. **25.** 127. **26.** $175 \times 9!$ **27.** 56; 126; 90. **28.** 17,153,136. **29.** 555,015.

CHAPTER XIII

Pages 220–221

1. $\frac{1}{221}$. **2.** $\frac{2}{7}, \frac{1}{7}, \frac{4}{7}$. **3.** $\frac{4}{91}, \frac{4}{455}, \frac{2}{91}, \frac{12}{91}, \frac{12}{91}, \frac{8}{91}$. **4.** $\frac{1}{220}, \frac{34}{55}$.
5. $\frac{1}{8}, \frac{3}{8}, \frac{1}{2}$. **6.** $\frac{13}{18}$. **7.** Seven, 5 to 1. **8.** $\frac{7}{72}$. **9.** $\frac{91}{216}$. **10.** $\frac{1}{4}$.
11. $\frac{8}{13}$. **12.** $5.43 **13.** $\frac{1}{4}, \frac{1}{28}$.

Pages 224–225

1. $\frac{4}{105}$. **2.** $\frac{1}{6}, \frac{5}{36}, \frac{25}{216}, \frac{125}{1296}$. **3.** $\frac{91}{216}$. **4.** $\frac{53}{64}$. **5.** $\frac{1}{3}, \frac{4}{15}, \frac{1}{5}, \frac{2}{15}$.
6. $\frac{2197}{20825}$. **7.** $\frac{135}{256}$.

Page 226

1. 0.85; 0.76; 0.63; 0.42; 0.16 **2.** 0.16 **3.** 0.50; 0.08 **4.** 0.23
5. 0.07; 0.16 **6.** 0.31

CHAPTER XIV

Page 228

1. 8. **2.** 11. **3.** -7. **4.** 22. **5.** 15. **6.** 1164. **7.** 0.

Page 232

1. 120; 720. **2.** Seven, three. **5.** -384. **6.** 15.

Pages 235–237

5. 2. **6.** 0. **7.** 35. **12.** $2(a-2)(a-4)$.
13. $-3(a-b)(a-1)(b-1)$. **14.** $2(x-y)(y-z)(z-x)$.
15. $-ab(a-b)(a-1)(b-1)$. **16.** $(a-b)(b-c)(c-a)(ab+bc+ca)$.
17. $-(a-b)(b-c)(c-a)(a+b+c)$. **19.** $x=4, x=2$. **20.** $x=-1$.

Pages 240–241

1. 9. **2.** -1196. **3.** 0. **4.** 14. **5.** $(b-a)^3$. **6.** 0.
7. $a=-2, b=4, c=1$. **8.** $x=1, y=0, z=0$.
9. $v=3, x=5, y=4, z=1$. **10.** $x=-2, y=3, z=4, u=-4$.

CHAPTER XV

Page 246

1. $\frac{3}{2(x-5)} - \frac{1}{2(x+3)}$. **2.** $\frac{9}{2x+3} + \frac{8}{3x-2}$. **3.** $\frac{5}{2x-1} - \frac{4}{3x-1}$.

4. $\frac{4}{1-2x} - \frac{3}{1-x}$. **5.** $\frac{2}{2x-1} - \frac{6}{3x+2} + \frac{4}{4x-3}$.

6. $\frac{5}{3(x-1)} - \frac{3}{2x} - \frac{1}{6(x+2)}$. **7.** $\frac{1}{x+1} + \frac{2x-3}{x^2+x+1}$.

8. $\frac{1}{2x-3} - \frac{2x-5}{4x^2+6x+9}$. **9.** $\frac{1}{x-4} - \frac{3x+6}{3x^2+2}$.

Page 248

1. $\frac{1}{(x+1)^2} - \frac{5}{(x+1)^3}$. **2.** $\frac{1}{(x-1)^2} + \frac{1}{(x-1)^3} + \frac{2}{(x-1)^4}$.

3. $\frac{2}{2x-3} + \frac{12}{(2x-3)^2} + \frac{18}{(2x-3)^3}$. **4.** $\frac{9}{x-1} - \frac{9}{x-2} + \frac{9}{(x-2)^2}$.

5. $\frac{1}{x^2} + \frac{1}{(x-1)^2}$. **6.** $\frac{1}{x^2} - \frac{1}{x^2+1}$. **7.** $2x+3+\frac{1}{x-1}+\frac{5}{3x+1}$.

8. $2x^2 + \frac{2}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \frac{2}{x-1}$.

CHAPTER XVI

The exercises in this chapter have been solved by five place logarithms where the results cannot be obtained directly from pages 318–322. Results obtained by four place logarithms may be somewhat different.

Page 251

1. \$46.88 **2.** \$433.33 **3.** 2 years. **4.** \$487.80; \$12.20
5. \$48.78; 5.85%.

Page 256

1. \$2560.17 **2.** \$3380.97 **3.** \$2845.73 **4.** \$4319.20 **5.** \$2871.60
6. \$5468.25 **7.** 5.09%. **8.** 8.16%. **9.** 5.8%.
10. 6.3% compounded annually.

Page 259

1. \$600.31 **2.** \$6613.20 **3.** \$1196.01 **5.** \$3352.50 **6.** \$8220.00

Pages 260–261

1. \$865.90 **2.** \$1216.64 **3.** \$2932.75; \$2227.18 **6.** \$294.03
7. \$2094.44 **9.** \$288.59 **10.** \$1347.78 **11.** \$4537.85 **12.** \$3745.35

CHAPTER XVII

Page 264

16. $4 - i$. **19.** $42 + 9i$. **20.** $4 + 6i$. **21.** $-24 - 10i$.

22. 25. **25.** $\frac{9}{5} + \frac{12i}{5}$. **26.** $1 - i$. **27.** $\frac{-6}{25} + \frac{17i}{25}$.

28. $\frac{6 + 4\sqrt{3}}{21} + \frac{(6 - 4\sqrt{3})i}{21}$. **29.** $\frac{10 - \sqrt{6}}{7} - \frac{(2\sqrt{2} + 5\sqrt{3})i}{7}$.

30. $\frac{2}{9} - \frac{i\sqrt{5}}{9}$. **32.** $0, -\frac{2}{3}$. **34.** $-2, -2$.

Page 266

1. $2[\cos(-60°) + i\sin(-60°)]$. **2.** $3\sqrt{2}(\cos 45° + i\sin 45°)$.
3. $2[\cos(-30°) + i\sin(-30°)]$. **4.** $2(\cos 30° + i\sin 30°)$.
5. $1(\cos 90° + i\sin 90°)$. **6.** $2\sqrt{2}(\cos 45° + i\sin 45°)$.
7. $3[\cos(-90°) + i\sin(-90°)]$. **8.** $\sqrt{17}(\cos 14° 3' + i\sin 14° 3')$.
9. $5(\cos 53° 4' + i\sin 53° 4')$. **10.** $13(\cos 22° 37' + i\sin 22° 37')$.
11. $\sqrt{85}(\cos 49° 24' + i\sin 49° 24')$.
12. $\sqrt{73}[\cos(-69° 27') + i\sin(-69° 27')]$. **13.** $\sqrt{3} + i$.
14. $-3/2 + 3i(\sqrt{3}/2)$. **15.** $-1.2940 + 4.8295i$. **16.** $3i$. **17.** -7.
18. $-5i$.

Page 268

1. $10(\cos 90° + i\sin 90°)$. **3.** $12(\cos 171° + i\sin 171°)$.
5. $\cos 45° + i\sin 45°$. **7.** $2[\cos(-180°) + i\sin(-180°)]$.
8. $4\sqrt{2}[\cos(-15°) + i\sin(-15°)]$. **9.** $\sqrt{2}(\cos 105° + i\sin 105°)$.

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4. $256(\cos 60° + i\sin 60°)$. **6.** $-8 + 8i\sqrt{3}$. **7.** $-16 - 16i$.

Pages 270–271

6. $\cos(-20° + k\cdot 120°) + i\sin(-20° + k\cdot 120°)$, where $k = 0, 1, 2$.
7. $\cos(45° + k\cdot 180°) + i\sin(45° + k\cdot 180°)$, where $k = 0, 1$.
9. $3[\cos(45° + k\cdot 90°) + i\sin(45° + k\cdot 90°)]$, where $k = 0, 1, 2, 3$.
11. $\cos(60° + k\cdot 120°) + i\sin(60° + k\cdot 120°)$, where $k = 0, 1, 2$.
13. $\cos(k\cdot 60°) + i\sin(k\cdot 60°)$, where $k = 0, 1, 2, 3, 4, 5$.
15. $2[\cos(k\cdot 90°) + i\sin(k\cdot 90°)]$, where $k = 0, 1, 2, 3$.

CHAPTER XVIII

Page 275

1. 83.	**2.** 1258.	**3.** 48.	**4.** 1033.	**5.** 5036.
6. 5335.	**7.** 9345.	**8.** 1028.	**9.** 30230.	**10.** 2251.

Page 277

1. 11525.	**2.** 12110.	**3.** 3325.	**4.** 4023.
5. 125211.	**6.** 421333.	**7.** 3.	**8.** 25.

TABLES

		Pages
	Logarithms—See pages 104–105.	
I.	Trigonometric Functions	299
II.	Powers—Roots—Reciprocals	300
III*a*.	Compound Interest	318
III*b*.	Compound Discount	319
III*c*.	Amount of an Annuity	320
III*d*.	Present Value of an Annuity	321
III*e*.	Logarithms for Interest Computations	322
III*f*.	American Experience Mortality Table	322

Logarithms. No table of logarithms is included here because a table is given on pp. 104–105. For a more complete table, see The Macmillan Tables, pp. 1–19.

I. Trigonometric Functions. This simple table requires no explanation if the ordinary trigonometric definitions are known. For a more complete table, see The Macmillan Tables, pp. 21–90, or pp. 138–142.

II. Powers—Roots—Reciprocals. This table is arranged so that the square, cube, square root, cube root, or reciprocal can be read directly to five decimal places for any number n of three significant figures. To attain this, not only n^2, n^3, $\sqrt{n}$, $\sqrt[3]{n}$, $1/n$, but also $\sqrt{10n}$, $\sqrt[3]{10n}$, $\sqrt[3]{100n}$ are printed on every page. All values have been carefully recomputed and checked.

Thus to find $\sqrt{1.17}$, read in $\sqrt{n}$ column the result: 1.08167 To find $\sqrt{11.7}$, read *in the same line*, in $\sqrt{10n}$ column the result: 3.42053 To find $\sqrt{117}$, read 10 times the entry in $\sqrt{n}$ column, since $\sqrt{117} = 10\sqrt{1.17}$

Similarly, $\sqrt[3]{1.17} = 1.05373$ from $\sqrt[3]{n}$ column; $\sqrt[3]{11.7} = 2.27019$ from *the same line* in $\sqrt[3]{10n}$ column; $\sqrt[3]{117} = 4.89097$ from *the same line* in $\sqrt[3]{100n}$ column.

The effect of a change in the decimal point in n^2, n^3, and $1/n$ is only to shift the decimal point in the result, without altering the digits printed.

III. Interest Tables—Mortality Table. Tables III*a*, *b*, *c*, *d* give compound interest and annuity data for various per cents up to fifty years. Aside from the obvious uses, formulas involving this data will be found in works on statistics, accounting, and the mathematics of business.

Table III*e* gives the logarithms of $(1 + r)$ to fifteen places, for all ordinary values of r from $\frac{1}{2}\%$ to 10%. For other values of r, $\log (1 + r)$ may be computed from any ordinary logarithm table.

Table III*f* is the American Experience Mortality Table.

[Characteristics of Logarithms omitted—determine by the usual rule from the value]

Radians	Degrees	Sine Value	Sine $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Cosine Value	Cosine $\log_{10}$		
0000	0°	.0000	−∞	.0000	−∞	∞	∞	1.0000	0000	90°	1.5708
.0175	1°	.0175	2419	.0175	2419	57.290	7581	.9998	9999	89°	1.5533
.0349	2°	.0349	5428	.0349	5431	28.636	4569	.9994	9997	88°	1.5359
.0524	3°	.0523	7188	.0524	7194	19.081	2806	.9986	9994	87°	1.5184
.0698	4°	.0698	8436	.0699	8446	14.301	1554	.9976	9989	86°	1.5010
.0873	5°	.0872	9403	.0875	9420	11.430	0580	.9962	9983	85°	1.4835
.1047	6°	.1045	0192	.1051	0216	9.5144	9784	.9945	9976	84°	1.4661
.1222	7°	.1219	0859	.1228	0891	8.1443	9109	.9925	9968	83°	1.4486
.1396	8°	.1392	1436	.1405	1478	7.1154	8522	.9903	9958	82°	1.4312
.1571	9°	.1564	1943	.1584	1997	6.3138	8003	.9877	9946	81°	1.4137
.1745	10°	.1736	2397	.1763	2463	5.6713	7537	.9848	9934	80°	1.3963
.1920	11°	.1908	2806	.1944	2887	5.1446	7113	.9816	9919	79°	1.3788
.2094	12°	.2079	3179	.2126	3275	4.7046	6725	.9781	9904	78°	1.3614
.2269	13°	.2250	3521	.2309	3634	4.3315	6366	.9744	9887	77°	1.3439
.2443	14°	.2419	3837	.2493	3968	4.0108	6032	.9703	9869	76°	1.3265
.2618	15°	.2588	4130	.2679	4281	3.7321	5719	.9659	9849	75°	1.3090
.2793	16°	.2756	4403	.2867	4575	3.4874	5425	.9613	9828	74°	1.2915
.2967	17°	.2924	4659	.3057	4853	3.2709	5147	.9563	9806	73°	1.2741
.3142	18°	.3090	4900	.3249	5118	3.0777	4882	.9511	9782	72°	1.2566
.3316	19°	.3256	5126	.3443	5370	2.9042	4630	.9455	9757	71°	1.2392
.3491	20°	.3420	5341	.3640	5611	2.7475	4389	.9397	9730	70°	1.2217
.3665	21°	.3584	5543	.3839	5842	2.6051	4158	.9336	9702	69°	1.2043
.3840	22°	.3746	5736	.4040	6064	2.4751	3936	.9272	9672	68°	1.1868
.4014	23°	.3907	5919	.4245	6279	2.3559	3721	.9205	9640	67°	1.1694
.4189	24°	.4067	6093	.4452	6486	2.2460	3514	.9135	9607	66°	1.1519
.4363	25°	.4226	6259	.4663	6687	2.1445	3313	.9063	9573	65°	1.1345
.4538	26°	.4384	6418	.4877	6882	2.0503	3118	.8988	9537	64°	1.1170
.4712	27°	.4540	6570	.5095	7072	1.9626	2928	.8910	9499	63°	1.0996
.4887	28°	.4695	6716	.5317	7257	1.8807	2743	.8829	9459	62°	1.0821
.5061	29°	.4848	6856	.5543	7438	1.8040	2562	.8746	9418	61°	1.0647
.5236	30°	.5000	6990	.5774	7614	1.7321	2386	.8660	9375	60°	1.0472
.5411	31°	.5150	7118	.6009	7788	1.6643	2212	.8572	9331	59°	1.0297
.5585	32°	.5299	7242	.6249	7958	1.6003	2042	.8480	9284	58°	1.0123
.5760	33°	.5446	7361	.6494	8125	1.5399	1875	.8387	9236	57°	.9948
.5934	34°	.5592	7476	.6745	8290	1.4826	1710	.8290	9186	56°	.9774
.6109	35°	.5736	7586	.7002	8452	1.4281	1548	.8192	9134	55°	.9599
.6283	36°	.5878	7692	.7265	8613	1.3764	1387	.8090	9080	54°	.9425
.6458	37°	.6018	7795	.7536	8771	1.3270	1229	.7986	9023	53°	.9250
.6632	38°	.6157	7893	.7813	8928	1.2799	1072	.7880	8965	52°	.9076
.6807	39°	.6293	7989	.8098	9084	1.2349	0916	.7771	8905	51°	.8901
.6981	40°	.6428	8081	.8391	9238	1.1918	0762	.7660	8843	50°	.8727
.7156	41°	.6561	8169	.8693	9392	1.1504	0608	.7547	8778	49°	.8552
.7330	42°	.6691	8255	.9004	9544	1.1106	0456	.7431	8711	48°	.8378
.7505	43°	.6820	8338	.9325	9697	1.0724	0303	.7314	8641	47°	.8203
.7679	44°	.6947	8418	.9657	9848	1.0355	0152	.7193	8569	46°	.8029
.7854	45°	.7071	8495	1.0000	0000	1.0000	0000	.7071	8495	45°	.7854
		Value	Cosine $\log_{10}$	Value	Cotangent $\log_{10}$	Value	Tangent $\log_{10}$	Value	Sine $\log_{10}$	Degrees	Radians

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$	$1/n$
1.00	1.0000	1.00000	3.16228	1.00000	1.00000	2.15443	4.64159	1.00000
1.01	1.0201	1.00499	3.17805	1.03030	1.00332	2.16159	4.65701	.990099
1.02	1.0404	1.00995	3.19374	1.06121	1.00662	2.16870	4.67233	.980392
1.03	1.0609	1.01489	3.20936	1.09273	1.00990	2.17577	4.68755	.970874
1.04	1.0816	1.01980	3.22490	1.12486	1.01316	2.18279	4.70267	.961538
1.05	1.1025	1.02470	3.24037	1.15762	1.01640	2.18976	4.71769	.952381
1.06	1.1236	1.02956	3.25576	1.19102	1.01961	2.19669	4.73262	.943396
1.07	1.1449	1.03441	3.27109	1.22504	1.02281	2.20358	4.74746	.934579
1.08	1.1664	1.03923	3.28634	1 25971	1.02599	2.21042	4.76220	.925926
1.09	1.1881	1.04403	3.30151	1.29503	1.02914	2.21722	4.77686	.917431
1.10	1.2100	1.04881	3.31662	1.33100	1.03228	2.22398	4.79142	.909091
1.11	1.2321	1.05357	3.33167	1.36763	1.03540	2.23070	4.80590	.900901
1.12	1.2544	1.05830	3.34664	1.40493	1.03850	2.23738	4.82028	.892857
1.13	1.2769	1.06301	3.36155	1.44290	1.04158	2.24402	4.83459	.884956
1.14	1.2996	1.06771	3.37639	1.48154	1.04464	2.25062	4.84881	.877193
1.15	1.3225	1.07238	3.39116	1.52088	1.04769	2.25718	4.86294	.869565
1.16	1.3456	1.07703	3.40588	1.56090	1.05072	2.26370	4.87700	.862069
1.17	1.3689	1.08167	3.42053	1.60161	1.05373	2.27019	4.89097	.854701
1.18	1.3924	1.08628	3.43511	1.64303	1.05672	2.27664	4.90487	.847458
1.19	1.4161	1.09087	3 44964	1.68516	1.05970	2.28305	4.91868	.840336
1.20	1.4400	1.09545	3.46410	1.72800	1.06266	2.28943	4.93242	.833333
1.21	1.4641	1.10000	3.47851	1.77156	1.06560	2.29577	4.94609	.826446
1.22	1.4884	1.10454	3.49285	1.81585	1.06853	2.30208	4.95968	.819672
1.23	1.5129	1.10905	3.50714	1.86087	1.07144	2.30835	4.97319	.813008
1.24	1.5376	1.11355	3.52136	1.90662	1.07434	2.31459	4.98663	.806452
1.25	1.5625	1.11803	3.53553	1.95312	1.07722	2.32079	5.00000	.800000
1.26	1.5876	1.12250	3.54965	2.00038	1.08008	2.32697	5.01330	.793651
1.27	1.6129	1.12694	3.56371	2.04838	1.08293	2.33311	5.02653	.787402
1.28	1.6384	1.13137	3.57771	2.09715	1.08577	2.33921	5.03968	.781250
1.29	1.6641	1.13578	3.59166	2.14669	1.08859	2.34529	5.05277	.775194
1.30	1.6900	1.14018	3.60555	2.19700	1.09139	2.35133	5.06580	.769231
1.31	1.7161	1.14455	3 61939	2.24809	1.09418	2.35735	5.07875	.763359
1.32	1.7424	1.14891	3.63318	2.29997	1.09696	2.36333	5.09164	.757576
1.33	1.7689	1.15326	3.64692	2.35264	1.09972	2.36928	5.10447	.751880
1.34	1.7956	1.15758	3.66060	2.40610	1.10247	2.37521	5.11723	.746269
1.35	1.8225	1.16190	3.67423	2.46038	1.10521	2.38110	5.12993	.740741
1.36	1.8496	1.16619	3.68782	2.51546	1.10793	2.38697	5.14256	.735294
1.37	1.8769	1.17047	3.70135	2.57135	1.11064	2.39280	5.15514	.729927
1.38	1.9044	1.17473	3.71484	2.62807	1.11334	2.39861	5.16765	.724638
1.39	1.9321	1.17898	3.72827	2.68562	1.11602	2.40439	5.18010	.719424
1 40	1.9600	1.18322	3.74166	2.74400	1.11869	2.41014	5.19249	.714286
1.41	1.9881	1.18743	3.75500	2.80322	1.12135	2.41587	5.20483	.709220
1.42	2.0164	1.19164	3.76829	2.86329	1.12399	2.42156	5.21710	.704225
1.43	2 0449	1.19583	3.78153	2.92421	1.12662	2.42724	5.22932	.699301
1.44	2.0736	1.20000	3.79473	2.98598	1.12924	2.43288	5.24148	.694444
1.45	2.1025	1.20416	3.80789	3.04862	1.13185	2.43850	5.25359	.689655
1.46	2.1316	1.20830	3.82099	3.11214	1.13445	2.44409	5.26564	.684932
1.47	2.1609	1.21244	3.83406	3.17652	1.13703	2.44966	5.27763	.680272
1.48	2.1904	1.21655	3.84708	3.24179	1.13960	2.45520	5.28957	.675676
1.49	2.2201	1.22066	3.86005	3.30795	1.14216	2.46072	5.30146	.671141
1.50	2.2500	1.22474	3.87298	3.37500	1.14471	2.46621	5.31329	.666667
n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$	$1/n$

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$	$1/n$
1 50	2.2500	1.22474	3.87298	3.37500	1.14471	2.46621	5.31329	.666667
1.51	2.2801	1.22882	3.88587	3.44295	1.14725	2.47168	5.32507	.662252
1.52	2.3104	1.23288	3.89872	3.51181	1.14978	2.47712	5.33680	.657895
1.53	2.3409	1.23693	3.91152	3.58158	1.15230	2.48255	5.34848	.653595
1.54	2.3716	1.24097	3.92428	3.65226	1.15480	2.48794	5.36011	.649351
1.55	2.4025	1.24499	3.93700	3.72388	1.15729	2.49332	5.37169	.645161
1.56	2.4336	1.24900	3.94968	3.79642	1.15978	2.49867	5.38321	.641026
1.57	2.4649	1.25300	3.96232	3.86989	1.16225	2.50399	5.39469	.636943
1.58	2.4964	1.25698	3.97492	3.94431	1.16471	2.50930	5.40612	.632911
1.59	2.5281	1.26095	3.98748	4.01968	1.16717	2.51458	5 41750	.628931
1 60	2.5600	1.26491	4.00000	4.09600	1.16961	2.51984	5 42884	.625000
1.61	2.5921	1.26886	4.01248	4.17328	1.17204	2.52508	5.44012	.621118
1.62	2.6244	1 27279	4.02492	4.25153	1.17446	2 53030	5 45136	.617284
1.63	2.6569	1.27671	4.03733	4.33075	1.17687	2.53549	5.46256	.613497
1.64	2.6896	1.28062	4.04969	4.41094	1.17927	2 54067	5.47370	.609756
1.65	2.7225	1.28452	4.06202	4.49212	1.18167	2.54582	5.48481	.606061
1.66	2.7556	1.28841	4.07431	4.57430	1.18405	2.55095	5.49586	.602410
1.67	2.7889	1.29228	4.08656	4.65746	1.18642	2.55607	5.50688	.598802
1.68	2.8224	1.29615	4.09878	4.74163	1.18878	2.56116	5.51785	.595238
1.69	2.8561	1.30000	4.11096	4.82681	1.19114	2.56623	5.52877	.591716
1.70	2.8900	1.30384	4.12311	4.91300	1.19348	2.57128	5.53966	.588235
1.71	2.9241	1.30767	4.13521	5.00021	1.19582	2.57631	5 55050	.584795
1.72	2.9584	1.31149	4.14729	5.08845	1.19815	2.58133	5.56130	.581395
1.73	2.9929	1.31529	4.15933	5.17772	1.20046	2.58632	5.57205	.578035
1.74	3.0276	1.31909	4.17133	5.26802	1.20277	2.59129	5.58277	.574713
1.75	3.0625	1.32288	4.18330	5.35938	1.20507	2.59625	5.59344	.571429
1.76	3.0976	1.32665	4.19524	5.45178	1.20736	2.60118	5.60408	.568182
1.77	3.1329	1.33041	4.20714	5.54523	1.20964	2.60610	5.61467	.564972
1.78	3.1684	1.33417	4.21900	5.63975	1.21192	2.61100	5.62523	.561798
1.79	3.2041	1.33791	4.23084	5.73534	1.21418	2.61588	5.63574	.558659
1.80	3.2400	1.34164	4 24264	5.83200	1.21644	2.62074	5.64622	.555556
1.81	3.2761	1.34536	4.25441	5.92974	1.21869	2.62559	5.65665	.552486
1 82	3.3124	1.34907	4.26615	6.02857	1.22093	2.63041	5.66705	.549451
1.83	3.3489	1.35277	4.27785	6.12849	1.22316	2.63522	5.67741	.546448
1.84	3.3856	1.35647	4.28952	6.22950	1.22539	2.64001	5.68773	.543478
1.85	3.4225	1.36015	4.30116	6.33162	1.22760	2.64479	5.69802	.540541
1.86	3.4596	1.36382	4.31277	6.43486	1.22981	2.64954	5.70827	.537634
1.87	3.4969	1.36748	4.32435	6.53920	1.23201	2.65428	5.71848	.534759
1.88	3.5344	1.37113	4.33590	6.64467	1.23420	2.65901	5.72865	.531915
1.89	3.5721	1.37477	4.34741	6.75127	1.23639	2.66371	5.73879	.529101
1.90	3.6100	1.37840	4.35890	6.85900	1.23856	2.66840	5.74890	.526316
1.91	3.6481	1.38203	4.37035	6.96787	1.24073	2 67307	5.75897	.523560
1.92	3.6864	1.38564	4.38178	7.07789	1.24289	2.67773	5.76900	.520833
1.93	3.7249	1.38924	4.39318	7.18906	1.24505	2.68237	5.77900	.518135
1.94	3.7636	1.39284	4.40454	7.30138	1.24719	2 68700	5.78896	.515464
1.95	3.8025	1.39642	4.41588	7.41488	1.24933	2.69161	5.79889	.512821
1.96	3.8416	1.40000	4.42719	7.52954	1.25146	2.69620	5.80879	.510204
1.97	3.8809	1.40357	4.43847	7.64537	1.25359	2.70078	5.81865	.507614
1.98	3.9204	1.40712	4.44972	7.76239	1.25571	2.70534	5.82848	.505051
1.99	3.9601	1.41067	4.46094	7.88060	1.25782	2.70989	5.83827	.502513
2.00	4.0000	1.41421	4.47214	8.00000	1.25992	2.71442	5.84804	.500000
n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$	$1/n$

n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$
2.00	4.0000	1.41421	4.47214	8.00000	1.25992	2.71442	5.84804	.500000
2.01	4.0401	1.41774	4 48330	8.12060	1.26202	2.71893	5.85777	.497512
2.02	4.0804	1.42127	4.49444	8.24241	1.26411	2.72344	5.86746	.495050
2.03	4.1209	1.42478	4.50555	8.36543	1.26619	2.72792	5.87713	.492611
2.04	4.1616	1.42829	4.51664	8.48966	1.26827	2.73239	5.88677	.490196
2.05	4 2025	1.43178	4.52769	8.61512	1.27033	2.73685	5.89637	.487805
2.06	4.2436	1.43527	4.53872	8.74182	1.27240	2.74129	5.90594	.485437
2.07	4.2849	1.43875	4.54973	8.86974	1.27445	2.74572	5.91548	.483092
2.08	4.3264	1.44222	4.56070	8.99891	1.27650	2.75014	5.92499	.480769
2.09	4.3681	1.44568	4.57165	9.12933	1.27854	2.75454	5.93447	.478469
2.10	4.4100	1.44914	4.58258	9 26100	1.28058	2.75892	5.94392	.476190
2.11	4.4521	1.45258	4.59347	9.39393	1.28261	2.76330	5.95334	.473934
2.12	4.4944	1.45602	4.60435	9.52813	1.28463	2.76766	5.96273	.471698
2.13	4.5369	1.45945	4.61519	9.66360	1.28665	2.77200	5.97209	.469434
2.14	4.5796	1.46287	4.62601	9.80034	1.28866	2.77633	5.98142	.467290
2.15	4.6225	1.46629	4.63681	9.93838	1.29066	2.78065	5.99073	.465116
2.16	4.6656	1.46969	4.64758	10.0777	1.29266	2.78495	6.00000	.462963
2.17	4.7089	1.47309	4.65833	10.2183	1.29465	2.78924	6.00925	.460829
2.18	4.7524	1.47648	4.66905	10.3602	1.29664	2.79352	6.01846	.458716
2.19	4.7961	1.47986	4.67974	10.5035	1.29862	2.79779	6.02765	.456621
2.20	4.8400	1.48324	4.69042	10.6480	1.30059	2.80204	6.03681	.454545
2.21	4.8841	1.48661	4.70106	10.7939	1.30256	2.80628	6.04594	.452489
2.22	4.9284	1.48997	4.71169	10.9410	1.30452	2.81050	6.05505	.450450
2.23	4.9729	1.49332	4.72229	11.0896	1.30648	2.81472	6.06413	.448430
2.24	5.0176	1.49666	4.73286	11.2394	1.30843	2.81892	6.07318	.446429
2.25	5.0625	1.50000	4.74342	11.3906	1.31037	2.82311	6.08220	.444444
2.26	5.1076	1.50333	4.75395	11.5432	1.31231	2.82728	6.09120	.442478
2.27	5.1529	1.50665	4.76445	11.6971	1.31424	2.83145	6.10017	.440529
2.28	5.1984	1.50997	4.77493	11.8524	1.31617	2.83560	6.10911	.438596
2.29	5.2441	1.51327	4.78539	12.0090	1.31809	2.83974	6.11803	.436681
2.30	5.2900	1 51658	4.79583	12.1670	1.32001	2.84387	6 12693	.434783
2.31	5.3361	1.51987	4.80625	12.3264	1.32192	2.84798	6.13579	.432900
2.32	5.3824	1.52315	4.81664	12.4872	1.32382	2.85209	6.14463	.431034
2.33	5.4289	1.52643	4.82701	12.6493	1.32572	2.85618	6.15345	.429185
2.34	5.4756	1.52971	4.83735	12.8129	1.32761	2.86026	6.16224	.427350
2.35	5.5225	1.53297	4.84768	12.9779	1.32950	2.86433	6.17101	.425532
2.36	5.5696	1.53623	4.85798	13.1443	1.33139	2.86838	6.17975	.423729
2.37	5.6169	1.53948	4.86826	13.3121	1.33326	2.87243	6.18846	.421941
2.38	5.6644	1.54272	4.87852	13.4813	1.33514	2.87646	6.19715	.420168
2.39	5.7121	1.54596	4.88876	13.6519	1.33700	2.88049	6.20582	.418410
2.40	5.7600	1 54919	4.89898	13.8240	1.33887	2.88450	6.21447	.416667
2.41	5.8081	1.55242	4.90918	13.9975	1.34072	2.88850	6.22308	.414938
2.42	5.8564	1.55563	4.91935	14.1725	1.34257	2.89249	6.23168	.413223
2.43	5.9049	1.55885	4.92950	14.3489	1.34442	2.89647	6.24025	.411523
2.44	5.9536	1.56205	4.93964	14.5268	1.34626	2.90044	6.24880	.409836
2.45	6.0025	1.56525	4.94975	14.7061	1.34810	2.90439	6.25732	.408163
2.46	6.0516	1.56844	4.95984	14.8869	1.34993	2.90834	6.26583	.406504
2.47	6.1009	1.57162	4.96991	15.0692	1.35176	2.91227	6.27431	.404858
2.48	6.1504	1.57480	4.97996	15.2530	1.35358	2.91620	6.28276	.403226
2 49	6.2001	1.57797	4.98999	15.4382	1.35540	2.92011	6.29119	.401606
2.50	6.2500	1.58114	5.00000	15.6250	1.35721	2.92402	6.29961	.400000
n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$

n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$
2.50	6.2500	1.58114	5.00000	15.6250	1.35721	2.92402	6.29961	.400000
2.51	6.3001	1.58430	5.00999	15.8133	1.35902	2.92791	6.30799	.398406
2.52	6.3504	1.58745	5.01996	16.0030	1.36082	2.93179	6.31636	.396825
2.53	6.4009	1.59060	5.02991	16.1943	1.36262	2.93567	6.32470	.395257
2.54	6.4516	1.59374	5.03984	16.3871	1.36441	2.93953	6.33303	.393701
2.55	6.5025	1.59687	5.04975	16.5814	1.36620	2.94338	6.34133	.392157
2.56	6.5536	1.60000	5.05964	16.7772	1.36798	2.94723	6.34960	.390625
2.57	6.6049	1.60312	5.06952	16.9746	1.36976	2.95106	6.35786	.389105
2.58	6.6564	1.60624	5.07937	17.1735	1.37153	2.95488	6.36610	.387597
2.59	6.7081	1.60935	5.08920	17.3740	1.37330	2.95869	6.37431	.386100
2.60	6.7600	1.61245	5.09902	17.5760	1.37507	2.96250	6.38250	.384615
2.61	6.8121	1.61555	5.10882	17.7796	1.37683	2.96629	6.39068	.383142
2.62	6.8644	1.61864	5.11859	17.9847	1.37859	2.97007	6.39883	.381679
2.63	6.9169	1.62173	5.12835	18.1914	1.38034	2.97385	6.40696	.380228
2.64	6.9696	1.62481	5.13809	18.3997	1.38208	2.97761	6.41507	.378788
2.65	7.0225	1.62788	5.14782	18.6096	1.38383	2.98137	6.42316	.377358
2.66	7.0756	1.63095	5.15752	18.8211	1.38557	2.98511	6.43123	.375940
2.67	7.1289	1.63401	5.16720	19.0342	1.38730	2.98885	6.43928	.374532
2.68	7.1824	1.63707	5.17687	19.2488	1.38903	2.99257	6.44731	.373134
2.69	7.2361	1.64012	5.18652	19.4651	1.39076	2.99629	6.45531	.371747
2.70	7.2900	1.64317	5.19615	19.6830	1.39248	3.00000	6.46330	.370370
2.71	7.3441	1.64621	5.20577	19.9025	1.39419	3.00370	6.47127	.369004
2.72	7.3984	1.64924	5.21536	20.1236	1.39591	3.00739	6.47922	.367647
2.73	7.4529	1.65227	5.22494	20.3464	1.39761	3.01107	6.48715	.366300
2.74	7.5076	1.65529	5.23450	20.5708	1.39932	3.01474	6.49507	.364964
2.75	7.5625	1.65831	5.24404	20.7969	1.40102	3.01841	6.50296	.363636
2.76	7.6176	1.66132	5.25357	21.0246	1.40272	3.02206	6.51083	.362319
2.77	7.6729	1.66433	5.26308	21.2539	1.40441	3.02570	6.51868	.361011
2.78	7.7284	1.66733	5.27257	21.4850	1.40610	3.02934	6.52652	.359712
2.79	7.7841	1.67033	5.28205	21.7176	1.40778	3.03297	6.53434	.358423
2.80	7.8400	1.67332	5.29150	21.9520	1.40946	3.03659	6.54213	.357143
2.81	7.8961	1.67631	5.30094	22.1880	1.41114	3.04020	6.54991	.355872
2.82	7.9524	1.67929	5.31037	22.4258	1.41281	3.04380	6.55767	.354610
2.83	8.0089	1.68226	5.31977	22.6652	1.41448	3.04740	6.56541	.353357
2.84	8.0656	1.68523	5.32917	22.9063	1.41614	3.05098	6.57314	.352113
2.85	8.1225	1.68819	5.33854	23.1491	1.41780	3.05456	6.58084	.350877
2.86	8.1796	1.69115	5.34790	23.3937	1.41946	3.05813	6.58853	.349650
2.87	8.2369	1.69411	5.35724	23.6399	1.42111	3.06169	6.59620	.348432
2.88	8.2944	1.69706	5.36656	23.8879	1.42276	3.06524	6.60385	.347222
2.89	8.3521	1.70000	5.37587	24.1376	1.42440	3.06878	6.61149	.346021
2.90	8.4100	1.70294	5.38516	24.3890	1.42604	3.07232	6.61911	.344828
2.91	8.4681	1.70587	5.39444	24.6422	1.42768	3.07584	6.62671	.343643
2.92	8.5264	1.70880	5.40370	24.8971	1.42931	3.07936	6.63429	.342466
2.93	8.5849	1.71172	5.41295	25.1538	1.43094	3.08287	6.64185	.341297
2.94	8.6436	1.71464	5.42218	25.4122	1.43257	3.08638	6.64940	.340136
2.95	8.7025	1.71756	5.43139	25.6724	1.43419	3.08987	6.65693	.338983
2.96	8.7616	1.72047	5.44059	25.9343	1.43581	3.09336	6.66444	.337838
2.97	8.8209	1.72337	5.44977	26.1981	1.43743	3.09684	6.67194	.336700
2.98	8.8804	1.72627	5.45894	26.4636	1.43904	3.10031	6.67942	.335570
2.99	8.9401	1.72916	5.46809	26.7309	1.44065	3.10378	6.68688	.334448
3.00	9.0000	1.73205	5.47723	27.0000	1.44225	3.10723	6.69433	.333333
n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$	$1/n$
3.00	9.0000	1.73205	5.47723	27.0000	1.44225	3.10723	6.69433	.333333
3.01	9.0601	1.73494	5.48635	27.2709	1.44385	3.11068	6.70176	.332226
3.02	9.1204	1.73781	5.49545	27.5436	1.44545	3.11412	6.70917	.331126
3.03	9.1809	1.74069	5.50454	27.8181	1.44704	3.11756	6.71657	.330033
3.04	9.2416	1.74356	5.51362	28.0945	1.44863	3.12098	6.72395	.328947
3.05	9.3025	1.74642	5.52268	28.3726	1.45022	3.12440	6.73132	.327869
3.06	9.3636	1.74929	5.53173	28.6526	1.45180	3.12781	6.73866	.326797
3.07	9.4249	1.75214	5.54076	28.9344	1.45338	3.13121	6.74600	.325733
3.08	9.4864	1.75499	5.54977	29.2181	1.45496	3.13461	6.75331	.324675
3.09	9.5481	1.75784	5.55878	29.5036	1.45653	3.13800	6.76061	.323625
3.10	9.6100	1.76068	5.56776	29.7910	1.45810	3.14138	6.76790	.322581
3.11	9.6721	1.76352	5.57674	30.0802	1.45967	3.14475	6.77517	.321543
3.12	9.7344	1.76635	5.58570	30.3713	1.46123	3.14812	6.78242	.320513
3.13	9.7969	1.76918	5.59464	30.6643	1.46279	3.15148	6.78966	.319489
3.14	9.8596	1.77200	5.60357	30.9591	1.46434	3.15483	6.79688	.318471
3.15	9.9225	1.77482	5.61249	31.2559	1.46590	3.15818	6.80409	.317460
3.16	9.9856	1.77764	5.62139	31.5545	1.46745	3.16152	6.81128	.316456
3.17	10.0489	1.78045	5.63028	31.8550	1.46899	3.16485	6.81846	.315457
3.18	10.1124	1.78326	5.63915	32.1574	1.47054	3.16817	6.82562	.314465
3.19	10.1761	1.78606	5.64801	32.4618	1.47208	3.17149	6.83277	.313480
3.20	10.2400	1.78885	5.65685	32.7680	1.47361	3.17480	6.83990	.312500
3.21	10.3041	1.79165	5.66569	33.0762	1.47515	3.17811	6.84702	.311526
3.22	10.3684	1.79444	5.67450	33.3862	1.47668	3.18140	6.85412	.310559
3.23	10.4329	1.79722	5.68331	33.6983	1.47820	3.18469	6.86121	.309598
3.24	10.4976	1.80000	5.69210	34.0122	1.47973	3.18798	6.86829	.308642
3.25	10.5625	1.80278	5.70088	34.3281	1.48125	3.19125	6.87534	.307692
3.26	10.6276	1.80555	5.70964	34.6460	1.48277	3.19452	6.88239	.306748
3.27	10.6929	1.80831	5.71839	34.9658	1.48428	3.19778	6.88942	.305810
3.28	10.7584	1.81108	5.72713	35.2876	1.48579	3.20104	6.89643	.304878
3.29	10.8241	1.81384	5.73585	35.6113	1.48730	3.20429	6.90344	.303951
3.30	10.8900	1.81659	5.74456	35.9370	1.48881	3.20753	6.91042	.303030
3.31	10.9561	1.81934	5.75326	36.2647	1.49031	3.21077	6.91740	.302115
3.32	11.0224	1.82209	5.76194	36.5944	1.49181	3.21400	6.92436	.301205
3.33	11.0889	1.82483	5.77062	36.9260	1.49330	3.21722	6.93130	.300300
3.34	11.1556	1.82757	5.77927	37.2597	1.49480	3.22044	6.93823	.299401
3.35	11.2225	1.83030	5.78792	37.5954	1.49629	3.22365	6.94515	.298507
3.36	11.2896	1.83303	5.79655	37.9331	1.49777	3.22686	6.95205	.297619
3.37	11.3569	1.83576	5.80517	38.2728	1.49926	3.23006	6.95894	.296736
3.38	11.4244	1.83848	5.81378	38.6145	1.50074	3.23325	6.96582	.295858
3.39	11.4921	1.84120	5.82237	38.9582	1.50222	3.23643	6.97268	.294985
3.40	11.5600	1.84391	5.83095	39.3040	1.50369	3.23961	6.97953	.294118
3.41	11.6281	1.84662	5.83952	39.6518	1.50517	3.24278	6.98637	.293255
3.42	11.6964	1.84932	5.84808	40.0017	1.50664	3.24595	6.99319	.292398
3.43	11.7649	1.85203	5.85662	40.3536	1.50810	3.24911	7.00000	.291545
3.44	11.8336	1.85172	5.86515	40.7076	1.50957	3.25227	7.00680	.290698
3.45	11.9025	1.85742	5.87367	41.0636	1.51103	3.25542	7.01358	.289855
3.46	11.9716	1.86011	5.88218	41.4217	1.51249	3.25856	7.02035	.289017
3.47	12.0409	1.86279	5.89067	41.7819	1.51394	3.26169	7.02711	.288184
3.48	12.1104	1.86548	5.89915	42.1442	1.51540	3.26482	7.03385	.287356
3.49	12.1801	1.86815	5.90762	42.5085	1.51685	3.26795	7.04058	.286533
3.50	12.2500	1.87083	5.91608	42.8750	1.51829	3.27107	7.04730	.285714
n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$	$1/n$

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$	$1/n$
3.50	12.2500	1.87083	5.91608	42.8750	1.51829	3.27107	7.04730	.285714
3.51	12.3201	1.87350	5.92453	43.2436	1.51974	3.27418	7.05400	.284900
3.52	12.3904	1.87617	5.93296	43.6142	1.52118	3.27729	7.06070	.284091
3.53	12.4609	1.87883	5.94138	43.9870	1.52262	3.28039	7.06738	.283286
3.54	12.5316	1.88149	5.94979	44.3619	1.52406	3.28348	7.07404	.282486
3.55	12.6025	1.88414	5.95819	44.7389	1.52549	3.28657	7.08070	.281690
3.56	12.6736	1.88680	5.96657	45.1180	1.52692	3.28965	7.08734	.280899
3.57	12.7449	1.88944	5.97495	45.4993	1.52835	3.29273	7.09397	.280112
3.58	12.8164	1.89209	5.98331	45.8827	1.52978	3.29580	7.10059	.279330
3.59	12.8881	1.89473	5.99166	46.2683	1.53120	3.29887	7.10719	.278552
3.60	12.9600	1.89737	6.00000	46.6560	1.53262	3.30193	7.11379	.277778
3.61	13.0321	1.90000	6.00833	47.0459	1.53404	3.30498	7.12037	.277008
3.62	13.1044	1.90263	6.01664	47.4379	1.53545	3.30803	7.12694	.276243
3.63	13.1769	1.90526	6.02495	47.8321	1.53686	3.31107	7.13349	.275482
3.64	13.2496	1.90788	6.03324	48.2285	1.53827	3.31411	7.14004	.274725
3.65	13.3225	1.91050	6.04152	48.6271	1.53968	3.31714	7.14657	.273973
3.66	13.3956	1.91311	6.04979	49.0279	1.54109	3.32017	7.15309	.273224
3.67	13.4689	1.91572	6.05805	49.4309	1.54249	3.32319	7.15960	.272480
3.68	13.5424	1.91833	6.06630	49.8360	1.54389	3.32621	7.16610	.271739
3.69	13.6161	1.92094	6.07454	50.2434	1.54529	3.32922	7.17258	.271003
3.70	13.6900	1.92354	6.08276	50.6530	1.54668	3.33222	7.17905	.270270
3.71	13.7641	1.92614	6.09098	51.0648	1.54807	3.33522	7.18552	.269542
3.72	13.8384	1.92873	6.09918	51.4788	1.54946	3.33822	7.19197	.268817
3.73	13.9129	1.93132	6.10737	51.8951	1.55085	3.34120	7.19840	.268097
3.74	13.9876	1.93391	6.11555	52.3136	1.55223	3.34419	7.20483	.267380
3.75	14.0625	1.93649	6.12372	52.7344	1.55362	3.34716	7.21125	.266667
3.76	14.1376	1.93907	6.13188	53.1574	1.55500	3.35014	7.21765	.265957
3.77	14.2129	1.94165	6.14003	53.5826	1.55637	3.35310	7.22405	.265252
3.78	14.2884	1.94422	6.14817	54.0102	1.55775	3.35607	7.23043	.264550
3.79	14.3641	1.94679	6.15630	54.4399	1.55912	3.35902	7.23680	.263852
3.80	14.4400	1.94936	6.16441	54.8720	1.56049	3.36198	7.24316	.263158
3.81	14.5161	1.95192	6.17252	55.3063	1.56186	3.36492	7.24950	.262467
3.82	14.5924	1.95448	6.18061	55.7430	1.56322	3.36786	7.25584	.261780
3.83	14.6689	1.95704	6.18870	56.1819	1.56459	3.37080	7.26217	.261097
3.84	14.7456	1.95959	6.19677	56.6231	1.56595	3.37373	7.26848	.260417
3.85	14.8225	1.96214	6.20484	57.0666	1.56731	3.37666	7.27479	.259740
3.86	14.8996	1.96469	6.21289	57.5125	1.56866	3.37958	7.28108	.259067
3.87	14.9769	1.96723	6.22093	57.9606	1.57001	3.38249	7.28736	.258398
3.88	15.0544	1.96977	6.22896	58.4111	1.57137	3.38540	7.29363	.257732
3.89	15.1321	1.97231	6.23699	58.8639	1.57271	3.38831	7.29989	.257069
3.90	15.2100	1.97484	6.24500	59.3190	1.57406	3.39121	7.30614	.256410
3.91	15.2881	1.97737	6.25300	59.7765	1.57541	3.39411	7.31238	.255754
3.92	15.3664	1.97990	6.26099	60.2363	1.57675	3.39700	7.31861	.255102
3.93	15.4449	1.98242	6.26897	60.6985	1.57809	3.39988	7.32483	.254453
3.94	15.5236	1.98494	6.27694	61.1630	1.57942	3.40277	7.33104	.253807
3.95	15.6025	1.98746	6.28490	61.6299	1.58076	3.40564	7.33723	.253165
3.96	15.6816	1.98997	6.29285	62.0991	1.58209	3.40851	7.34342	.252525
3.97	15.7609	1.99249	6.30079	62.5708	1.58342	3.41138	7.34960	.251889
3.98	15.8404	1.99499	6.30872	63.0448	1.58475	3.41424	7.35576	.251256
3.99	15.9201	1.99750	6.31664	63.5212	1.58608	3.41710	7.36192	.250627
4.00	16.0000	2.00000	6.32456	64.0000	1.58740	3.41995	7.36806	.250000
n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$	$1/n$

n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$
4.00	16.0000	2.00000	6.32456	64.0000	1.58740	3.41995	7.36806	.250000
4.01	16.0801	2.00250	6.33246	64.4812	1.58872	3.42280	7.37420	.249377
4.02	16.1604	2.00499	6.34035	64.9648	1.59004	3.42564	7.38032	.248756
4.03	16.2409	2.00749	6.34823	65.4508	1.59136	3.42848	7.38644	.248139
4.04	16.3216	2.00998	6.35610	65.9393	1.59267	3.43131	7.39254	.247525
4.05	16.4025	2.01246	6.36396	66.4301	1.59399	3.43414	7.39864	.246914
4.06	16.4836	2.01494	6.37181	66.9234	1.59530	3.43697	7.40472	.246305
4.07	16.5649	2.01742	6.37966	67.4191	1.59661	3.43979	7.41080	.245700
4.08	16.6464	2.01990	6.38749	67.9173	1.59791	3.44260	7.41686	.245098
4.09	16.7281	2.02237	6.39531	68.4179	1.59922	3.44541	7.42291	.244499
4.10	16.8100	2.02485	6.40312	68.9210	1.60052	3.44822	7.42896	.243902
4.11	16.8921	2.02731	6.41093	69.4265	1.60182	3.45102	7.43499	.243309
4.12	16.9744	2.02978	6.41872	69.9345	1.60312	3.45382	7.44102	.242718
4.13	17.0569	2.03224	6.42651	70.4450	1.60441	3.45661	7.44703	.242131
4.14	17.1396	2.03470	6.43428	70.9579	1.60571	3.45939	7.45304	.241546
4.15	17.2225	2.03715	6.44205	71.4734	1.60700	3.46218	7.45904	.240964
4.16	17.3056	2.03961	6.44981	71.9913	1.60829	3.46496	7.46502	.240385
4.17	17.3889	2.04206	6.45755	72.5117	1.60958	3.46773	7.47100	.239808
4.18	17.4724	2.04450	6.46529	73.0346	1.61086	3.47050	7.47697	.239234
4.19	17.5561	2.04695	6.47302	73.5601	1.61215	3.47327	7.48292	.238663
4.20	17.6400	2.04939	6.48074	74.0880	1.61343	3.47603	7.48887	.238095
4.21	17.7241	2.05183	6.48845	74.6185	1.61471	3.47878	7.49481	.237530
4.22	17.8084	2.05426	6.49615	75.1514	1.61599	3.48154	7.50074	.236967
4.23	17.8929	2.05670	6.50384	75.6870	1.61726	3.48428	7.50666	.236407
4.24	17.9776	2.05913	6.51153	76.2250	1.61853	3.48703	7.51257	.235849
4.25	18.0625	2.06155	6.51920	76.7656	1.61981	3.48977	7.51847	.235294
4.26	18.1476	2.06398	6.52687	77.3088	1.62108	3.49250	7.52437	.234742
4.27	18.2329	2.06640	6.53452	77.8545	1.62234	3.49523	7.53025	.234192
4.28	18.3184	2.06882	6.54217	78.4028	1.62361	3.49796	7.53612	.233645
4.29	18.4041	2.07123	6.54981	78.9536	1.62487	3.50068	7.54199	.233100
4.30	18.4900	2.07364	6.55744	79.5070	1.62613	3.50340	7.54784	.232558
4.31	18.5761	2.07605	6.56506	80.0630	1.62739	3.50611	7.55369	.232019
4.32	18.6624	2.07846	6.57267	80.6216	1.62865	3.50882	7.55953	.231481
4.33	18.7489	2.08087	6.58027	81.1827	1.62991	3.51153	7.56535	.230947
4.34	18.8356	2.08327	6.58787	81.7465	1.63116	3.51423	7.57117	.230415
4.35	18.9225	2.08567	6.59545	82.3129	1.63241	3.51692	7.57698	.229885
4.36	19.0096	2.08806	6.60303	82.8819	1.63366	3.51962	7.58279	.229358
4.37	19.0969	2.09045	6.61060	83.4535	1.63491	3.52231	7.58858	.228833
4.38	19.1844	2.09284	6.61816	84.0277	1.63619	3.52499	7.59436	.228311
4.39	19.2721	2.09523	6.62571	84.6045	1.63740	3.52767	7.60014	.227790
4.40	19.3600	2.09762	6.63325	85.1840	1.63864	3.53035	7.60590	.227273
4.41	19.4481	2.10000	6.64078	85.7661	1.63988	3.53302	7.61166	.226757
4.42	19.5364	2.10238	6.64831	86.3509	1.64112	3.53569	7.61741	.226244
4.43	19.6249	2.10476	6.65582	86.9383	1.64236	3.53835	7.62315	.225734
4.44	19.7136	2.10713	6.66333	87.5284	1.64359	3.54101	7.62888	.225225
4.45	19.8025	2.10950	6.67083	88.1211	1.64483	3.54367	7.63461	.224719
4.46	19.8916	2.11187	6.67832	88.7165	1.64606	3.54632	7.64032	.224215
4.47	19.9809	2.11424	6.68581	89.3146	1.64729	3.54897	7.64603	.223714
4.48	20.0704	2.11660	6.69328	89.9154	1.64851	3.55162	7.65172	.223214
4.49	20.1601	2.11896	6.70075	90.5188	1.64974	3.55426	7.65741	.222717
4.50	20.2500	2.12132	6.70820	91.1250	1.65096	3.55689	7.66309	.222222
n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$

n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$
4.50	20.2500	2.12132	6.70820	91.1250	1.65096	3.55689	7.66309	.222222
4.51	20.3401	2.12368	6.71565	91.7339	1.65219	3.55953	7.66877	.221729
4.52	20.4304	2.12603	6.72309	92.3454	1.65341	3.56215	7 67443	.221239
4.53	20.5209	2.12838	6.73053	92.9597	1.65462	3.56478	7.68009	220751
4.54	20.6116	2.13073	6.73795	93.5767	1.65584	3.56740	7.68573	.220264
4.55	20.7025	2.13307	6.74537	94.1964	1.65706	3.57002	7.69137	.219780
4.56	20.7936	2.13542	6.75278	94.8188	1.65827	3.57263	7.69700	.219298
4.57	20.8849	2.13776	6.76018	95.4440	1.65948	3.57524	7 70262	.218818
4.58	20.9764	2.14009	6.76757	96.0719	1.66069	3.57785	7 70824	.218341
4.59	21.0681	2.14243	6.77495	96.7026	1.66190	3.58045	7 71384	.217865
4.60	21.1600	2.14476	6.78233	97.3360	1.66310	3.58305	7.71944	.217391
4.61	21.2521	2.14709	6.78970	97.9722	1.66431	3.58564	7.72503	.216920
4.62	21.3444	2.14942	6.79706	98.6111	1.66551	3.58823	7 73061	.216450
4.63	21.4369	2.15174	6.80441	99.2528	1.66671	3.59082	7.73619	.215983
4.64	21.5296	2.15407	6.81175	99.8973	1.66791	3.59340	7.74175	.215517
4.65	21.6225	2.15639	6.81909	100.545	1.66911	3.59598	7 74731	215054
4.66	21.7156	2.15870	6.82642	101.195	1.67030	3.59856	7 75286	.214592
4.67	21.8089	2.16102	6.83374	101.848	1.67150	3.60113	7.75840	.214133
4.68	21.9024	2.16333	6.84105	102.503	1.67269	3.60370	7 76394	.213675
4.69	21.9961	2.16564	6.84836	103.162	1.67388	3.60626	7.76946	.213220
4.70	22.0900	2.16795	6.85565	103.823	1.67507	3.60883	7.77498	.212766
4.71	22.1841	2.17025	6.86294	104.487	1.67626	3.61138	7.78049	.212314
4.72	22.2784	2.17256	6.87023	105.154	1.67744	3.61394	7.78599	.211864
4.73	22.3729	2.17486	6.87750	105.824	1.67863	3.61649	7 79149	.211416
4.74	22.4676	2.17715	6.88477	106.496	1.67981	3.61903	7.79697	.210970
4.75	22.5625	2.17945	6.89202	107.172	1.68099	3.62158	7.80245	.210526
4.76	22.6576	2.18174	6.89928	107.850	1.68217	3.62412	7.80793	.210084
4.77	22.7529	2.18403	6.90652	108.531	1.68334	3.62665	7.81339	.209644
4.78	22.8484	2.18632	6.91375	109.215	1.68452	3.62919	7.81885	.209205
4.79	22.9441	2.18861	6.92098	109.902	1.68569	3.63172	7.82429	.208768
4.80	23.0400	2.19089	6.92820	110.592	1.68687	3.63424	7.82974	.208333
4.81	23.1361	2.19317	6.93542	111.285	1.68804	3.63676	7.83517	.207900
4.82	23.2324	2.19545	6.94262	111.980	1.68920	3.63928	7.84059	.207469
4.83	23.3289	2.19773	6.94982	112.679	1.69037	3.64180	7.84601	.207039
4.84	23.4256	2.20000	6.95701	113.380	1.69154	3.64431	7.85142	.206612
4.85	23.5225	2.20227	6.96419	114.084	1.69270	3.64682	7.85683	.206186
4.86	23.6196	2.20454	6.97137	114.791	1.69386	3.64932	7.86222	205761
4.87	23.7169	2.20681	6.97854	115.501	1.69503	3.65182	7.86761	.205339
4.88	23.8144	2.20907	6.98570	116.214	1.69619	3.65432	7.87299	.204918
4.89	23.9121	2.21133	6.99285	116.930	1.69734	3.65681	7.87837	.204499
4.90	24.0100	2.21359	7.00000	117.649	1.69850	3.65931	7.88374	.204082
4.91	24.1081	2.21585	7.00714	118.371	1.69965	3.66179	7.88909	.203666
4.92	24.2064	2.21811	7.01427	119.095	1.70081	3.66428	7.89445	.203252
4.93	24.3049	2.22036	7.02140	119.823	1.70196	3.66676	7.89979	.202840
4.94	24.4036	2.22261	7.02851	120.554	1.70311	3.66924	7.90513	.202429
4.95	24.5025	2.22486	7.03562	121.287	1.70426	3.67171	7.91046	.202020
4.96	24.6016	2.22711	7.04273	122.024	1.70540	3.67418	7.91578	.201613
4.97	24.7009	2.22935	7.04982	122.763	1.70655	3.67665	7.92110	.201207
4.98	24.8004	2.23159	7.05691	123.506	1.70769	3.67911	7.92641	.200803
4.99	24.9001	2.23383	7.06399	124.251	1.70884	3.68157	7.93171	.200401
5.00	25.0000	2.23607	7.07107	125.000	1.70998	3.68403	7.93701	.200000
n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$

n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$
5.00	25.0000	2.23607	7.07107	125.000	1.70998	3.68403	7.93701	.200000
5.01	25.1001	2.23830	7.07814	125.752	1.71112	3.68649	7.94229	.199601
5.02	25.2004	2.24054	7.08520	126.506	1.71225	3.68894	7.94757	.199203
5.03	25.3009	2.24277	7.09225	127.264	1.71339	3.69138	7.95285	.198807
5.04	25.4016	2.24499	7.09930	128.024	1.71452	3.69383	7.95811	.198413
5.05	25.5025	2.24722	7.10634	128.788	1.71566	3.69627	7.96337	.198020
5.06	25.6036	2.24944	7.11337	129.554	1.71679	3.69871	7.96863	.197628
5.07	25.7049	2.25167	7.12039	130.324	1.71792	3.70114	7.97387	.197239
5.08	25.8064	2.25389	7.12741	131.097	1.71905	3.70357	7.97911	.196850
5.09	25.9081	2.25610	7.13442	131.872	1.72017	3.70600	7.98434	.196464
5.10	26.0100	2.25832	7.14143	132.651	1.72130	3.70843	7.98957	.196078
5.11	26.1121	2.26053	7.14843	133.433	1.72242	3.71085	7.99479	.195695
5.12	26.2144	2.26274	7.15542	134.218	1.72355	3.71327	8.00000	.195312
5.13	26.3169	2.26495	7.16240	135.006	1.72467	3.71569	8.00520	.194932
5.14	26.4196	2.26716	7.16938	135.797	1.72579	3.71810	8.01040	.194553
5.15	26.5225	2.26936	7.17635	136.591	1.72691	3.72051	8.01559	.194175
5.16	26.6256	2.27156	7.18331	137.388	1.72802	3.72292	8.02078	.193798
5.17	26.7289	2.27376	7.19027	138.188	1.72914	3.72532	8.02596	.193424
5.18	26.8324	2.27596	7.19722	138.991	1.73025	3.72772	8.03113	.193050
5.19	26.9361	2.27816	7.20417	139.798	1.73137	3.73012	8.03629	.192678
5.20	27.0400	2.28035	7.21110	140.608	1.73248	3.73251	8.04145	.192308
5.21	27.1441	2.28254	7.21803	141.421	1.73359	3.73490	8.04660	.191939
5.22	27.2484	2.28473	7.22496	142.237	1.73470	3.73729	8.05175	.191571
5.23	27.3529	2.28692	7.23187	143.056	1.73580	3.73968	8.05689	.191205
5.24	27.4576	2.28910	7.23878	143.878	1.73691	3.74206	8.06202	.190840
5.25	27.5625	2.29129	7.24569	144.703	1.73801	3.74443	8.06714	.190476
5.26	27.6676	2.29347	7.25259	145.532	1.73912	3.74681	8.07226	.190114
5.27	27.7729	2.29565	7.25948	146.363	1.74022	3.74918	8.07737	.189753
5.28	27.8784	2.29783	7.26636	147.198	1.74132	3.75155	8.08248	.189394
5.29	27.9841	2.30000	7.27324	148.036	1.74242	3.75392	8.08758	.189036
5.30	28.0900	2.30217	7.28011	148.877	1.74351	3.75629	8.09267	.188679
5.31	28.1961	2.30434	7.28697	149.721	1.74461	3.75865	8.09776	.188324
5.32	28.3024	2.30651	7.29383	150.569	1.74570	3.76101	8.10284	.187970
5.33	28.4089	2.30868	7.30068	151.419	1.74680	3.76336	8.10791	.187617
5.34	28.5156	2.31084	7.30753	152.273	1.74789	3.76571	8.11298	.187266
5.35	28.6225	2.31301	7.31437	153.130	1.74898	3.76806	8.11804	.186916
5.36	28.7296	2.31517	7.32120	153.991	1.75007	3.77041	8.12310	.186567
5.37	28.8369	2.31733	7.32803	154.854	1.75116	3.77275	8.12814	.186220
5.38	28.9444	2.31948	7.33485	155.721	1.75224	3.77509	8.13319	.185874
5.39	29.0521	2.32164	7.34166	156.591	1.75333	3.77743	8.13822	.185529
5.40	29.1600	2.32379	7.34847	157.464	1.75441	3.77976	8.14325	.185185
5.41	29.2681	2.32594	7.35527	158.340	1.75549	3.78209	8.14828	.184843
5.42	29.3764	2.32809	7.36206	159.220	1.75657	3.78442	8.15329	.184502
5.43	29.4849	2.33024	7.36885	160.103	1.75765	3.78675	8.15831	.184162
5.44	29.5936	2.33238	7.37564	160.989	1.75873	3.78907	8.16331	.183824
5.45	29.7025	2.33452	7.38241	161.879	1.75981	3.79139	8.16831	.183486
5.46	29.8116	2.33666	7.38918	162.771	1.76088	3.79371	8.17330	.183150
5.47	29.9209	2.33880	7.39594	163.667	1.76196	3.79603	8.17829	.182815
5.48	30.0304	2.34094	7.40270	164.567	1.76303	3.79834	8.18327	.182482
5.49	30.1401	2.34307	7.40945	165.469	1.76410	3.80065	8.18824	.182149
5.50	30.2500	2.34521	7.41620	166.375	1.76517	3.80295	8.19321	.181818
n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$

n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$
5.50	30.2500	2.34521	7.41620	166.375	1.76517	3.80295	8.19321	.181818
5.51	30.3601	2.34734	7.42294	167.284	1.76624	3.80526	8.19818	.181488
5.52	30.4704	2.34947	7.42967	168.197	1.76731	3.80756	8.20313	.181159
5.53	30.5809	2.35160	7.43640	169.112	1.76838	3.80985	8.20808	.180832
5.54	30.6916	2.35372	7.44312	170.031	1.76944	3.81215	8.21303	.180505
5.55	30.8025	2.35584	7.44983	170.954	1.77051	3.81444	8.21797	.180180
5.56	30.9136	2.35797	7.45654	171.880	1.77157	3.81673	8.22290	.179856
5.57	31.0249	2.36008	7.46324	172.809	1.77263	3.81902	8.22783	.179533
5.58	31.1364	2.36220	7.46994	173.741	1.77369	3.82130	8.23275	.179211
5.59	31.2481	2.36432	7.47663	174.677	1.77475	3.82358	8.23766	.178891
5.60	31.3600	2.36643	7.48331	175.616	1.77581	3.82586	8.24257	.178571
5.61	31.4721	2.36854	7.48999	176.558	1.77686	3.82814	8.24747	.178253
5.62	31.5844	2.37065	7.49667	177.504	1.77792	3.83041	8.25237	.177936
5.63	31.6969	2.37276	7.50333	178.454	1.77897	3.83268	8.25726	.177620
5.64	31.8096	2.37487	7.50999	179.406	1.78003	3.83495	8.26215	.177305
5.65	31.9225	2.37697	7.51665	180.362	1.78108	3.83722	8.26703	.176991
5.66	32.0356	2.37908	7.52330	181.321	1.78213	3.83948	8.27190	.176678
5.67	32.1489	2.38118	7.52994	182.284	1.78318	3.84174	8.27677	.176367
5.68	32.2624	2.38328	7.53658	183.250	1.78422	3.84399	8.28164	.176056
5.69	32.3761	2.38537	7.54321	184.220	1.78527	3.84625	8.28649	.175747
5.70	32.4900	2.38747	7.54983	185.193	1.78632	3.84850	8.29134	.175439
5.71	32.6041	2.38956	7.55645	186.169	1.78736	3.85075	8.29619	.175131
5.72	32.7184	2.39165	7.56307	187.149	1.78840	3.85300	8.30103	.174825
5.73	32.8329	2.39374	7.56968	188.133	1.78944	3.85524	8.30587	.174520
5.74	32.9476	2.39583	7.57628	189.119	1.79048	3.85748	8.31069	.174216
5.75	33.0625	2.39792	7.58288	190.109	1.79152	3.85972	8.31552	.173913
5.76	33.1776	2.40000	7.58947	191.103	1.79256	3.86196	8.32034	.173611
5.77	33.2929	2.40208	7.59605	192.100	1.79360	3.86419	8.32515	.173310
5.78	33.4084	2.40416	7.60263	193.101	1.79463	3.86642	8.32995	.173010
5.79	33.5241	2.40624	7.60920	194.105	1.79567	3.86865	8.33476	.172712
5.80	33.6400	2.40832	7.61577	195.112	1.79670	3.87088	8.33955	.172414
5.81	33.7561	2.41039	7.62234	196.123	1.79773	3.87310	8.34434	.172117
5.82	33.8724	2.41247	7.62889	197.137	1.79876	3.87532	8.34913	.171821
5.83	33.9889	2.41454	7.63544	198.155	1.79979	3.87754	8.35390	.171527
5.84	34.1056	2.41661	7.64199	199.177	1.80082	3.87975	8.35868	.171233
5.85	34.2225	2.41868	7.64853	200.202	1.80185	3.88197	8.36345	.170940
5.86	34.3396	2.42074	7.65506	201.230	1.80288	3.88418	8.36821	.170649
5.87	34.4569	2.42281	7.66159	202.262	1.80390	3.88639	8.37297	.170358
5.88	34.5744	2.42487	7.66812	203.297	1.80492	3.88859	8.37772	.170068
5.89	34.6921	2.42693	7.67463	204.336	1.80595	3.89080	8.38247	.169779
5.90	34.8100	2.42899	7.68115	205.379	1.80697	3.89300	8.38721	.169492
5.91	34.9281	2.43105	7.68765	206.425	1.80799	3.89519	8.39194	.169205
5.92	35.0464	2.43311	7.69415	207.475	1.80901	3.89739	8.39667	.168919
5.93	35.1649	2.43516	7.70065	208.528	1.81003	3.89958	8.40140	.168634
5.94	35.2836	2.43721	7.70714	209.585	1.81104	3.90177	8.40612	.168350
5.95	35.4025	2.43926	7.71362	210.645	1.81206	3.90396	8.41083	.168067
5.96	35.5216	2.44131	7.72010	211.709	1.81307	3.90615	8.41554	.167785
5.97	35.6409	2.44336	7.72658	212.776	1.81409	3.90833	8.42025	.167504
5.98	35.7604	2.44540	7.73305	213.847	1.81510	3.91051	8.42494	.167224
5.99	35.8801	2.44745	7.73951	214.922	1.81611	3.91269	8.42964	.166945
6.00	36.0000	2.44949	7.74597	216.000	1.81712	3.91487	8.43433	.166667
n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$

n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$
6.00	36.0000	2.44949	7.74597	216.000	1.81712	3.91487	8.43433	.166667
6.01	36.1201	2.45153	7.75242	217.082	1.81813	3.91704	8.43901	.166389
6.02	36.2404	2.45357	7.75887	218.167	1.81914	3.91921	8.44369	.166113
6.03	36.3609	2.45561	7.76531	219.256	1.82014	3.92138	8.44836	.165837
6.04	36.4816	2.45764	7.77174	220.349	1.82115	3.92355	8.45303	.165563
6.05	36.6025	2.45967	7.77817	221.445	1.82215	3.92571	8.45769	.165289
6.06	36.7236	2.46171	7.78460	222.545	1.82316	3.92787	8.46235	.165017
6.07	36.8449	2.46374	7 79102	223.649	1.82416	3.93003	8.46700	.164745
6 08	36.9664	2.46577	7.79744	224.756	1.82516	3.93219	8.47165	.164474
6.09	37.0881	2.46779	7.80385	225.867	1.82616	3.93434	8.47629	.164204
6.10	37.2100	2.46982	7.81025	226.981	1.82716	3.93650	8.48093	.163934
6.11	37.3321	2.47184	7.81665	228.099	1.82816	3 93865	8.48556	.163666
6.12	37.4544	2.47386	7.82304	229.221	1.82915	3.94079	8.49018	.163399
6.13	37.5769	2.47588	7.82943	230.346	1.83015	3.94294	8.49481	.163132
6.14	37.6996	2.47790	7.83582	231.476	1.83115	3.94508	8.49942	.162866
6.15	37.8225	2.47992	7.84219	232.608	1.83214	3.94722	8.50403	.162602
6.16	37.9456	2.48193	7.84857	233.745	1.83313	3.94936	8.50864	.162338
6.17	38.0689	2.48395	7.85493	234.885	1.83412	3.95150	8.51324	.162075
6.18	38.1924	2.48596	7.86130	236.029	1.83511	3.95363	8.51784	.161812
6.19	38.3161	2.48797	7.86766	237.177	1.83610	3.95576	8.52243	.161551
6.20	38.4400	2.48998	7.87401	238.328	1.83709	3.95789	8.52702	.161290
6.21	38.5641	2.49199	7.88036	239.483	1.83808	3.96002	8.53160	.161031
6.22	38.6884	2.49399	7.88670	240.642	1.83906	3.96214	8.53618	.160772
6.23	38.8129	2.49600	7.89303	241.804	1.84005	3.96427	8.54075	.160514
6.24	38.9376	2.49800	7.89937	242.971	1.84103	3.96638	8.54532	.160256
6.25	39.0625	2.50000	7.90569	244.141	1.84202	3.96850	8.54988	.160000
6.26	39.1876	2.50200	7.91202	245.314	1.84300	3.97062	8.55444	.159744
6.27	39.3129	2.50400	7.91833	246.492	1.84398	3.97273	8.55899	.159490
6.28	39.4384	2.50599	7.92465	247.673	1.84496	3.97484	8.56354	.159236
6.29	39.5641	2.50799	7.93095	248.858	1.84594	3.97695	8.56808	.158983
6.30	39.6900	2.50998	7.93725	250.047	1.84691	3.97906	8.57262	.158730
6.31	39.8161	2.51197	7.94355	251.240	1.84789	3.98116	8.57715	.158479
6.32	39.9424	2.51396	7.94984	252.436	1.84887	3.98326	8.58168	.158228
6.33	40.0689	2.51595	7.95613	253.636	1.84984	3.98536	8.58620	.157978
6.34	40.1956	2.51794	7.96241	254.840	1.85082	3.98746	8.59072	.157729
6.35	40.3225	2.51992	7.96869	256.048	1.85179	3.98956	8.59524	.157480
6.36	40.4496	2.52190	7.97496	257.259	1.85276	3.99165	8.59975	.157233
6.37	40.5769	2.52389	7.98123	258.475	1.85373	3.99374	8.60425	.156986
6.38	40.7044	2.52587	7.98749	259.694	1.85470	3.99583	8.60875	.156740
6.39	40.8321	2.52784	7.99375	260.917	1.85567	3.99792	8.61325	.156495
6.40	40.9600	2.52982	8.00000	262.144	1.85664	4.00000	8.61774	.156250
6.41	41.0881	2.53180	8.00625	263.375	1.85760	4.00208	8.62222	.156006
6.42	41.2164	2.53377	8.01249	264.609	1.85857	4.00416	8.62671	.155763
6.43	41.3449	2.53574	8.01873	265.848	1.85953	4.00624	8.63118	.155521
6.44	41.4736	2.53772	8.02496	267.090	1.86050	4.00832	8.63566	.155280
6.45	41.6025	2.53969	8.03119	268.336	1.86146	4.01039	8.64012	.155039
6.46	41.7316	2.54165	8.03741	269.586	1.86242	4.01246	8.64459	.154799
6.47	41.8609	2.54362	8.04363	270.840	1.86338	4.01453	8.64904	.154560
6.48	41.9904	2.54558	8.04984	272.098	1.86434	4.01660	8.65350	.154321
6.49	42.1201	2.54755	8.05605	273.359	1.86530	4.01866	8.65795	.154083
6.50	42.2500	2.54951	8.06226	274.625	1.86626	4.02073	8.66239	.153846
n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$

n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$
6.50	42.2500	2.54951	8.06226	274.625	1.86626	4.02073	8.66239	.153846
6.51	42.3801	2.55147	8.06846	275.894	1.86721	4.02279	8.66683	.153610
6.52	42.5104	2.55343	8.07465	277.168	1.86817	4.02485	8.67127	.153374
6.53	42.6409	2.55539	8.08084	278.445	1.86912	4.02690	8.67570	.153139
6.54	42.7716	2.55734	8.08703	279.726	1.87008	4.02896	8.68012	.152905
6.55	42.9025	2.55930	8.09321	281.011	1.87103	4.03101	8.68455	.152672
6.56	43.0336	2.56125	8.09938	282.300	1.87198	4.03306	8.68896	.152439
6.57	43.1649	2.56320	8.10555	283.593	1.87293	4.03511	8.69338	.152207
6.58	43.2964	2.56515	8.11172	284.890	1.87388	4.03715	8.69778	.151976
6.59	43.4281	2.56710	8.11788	286.191	1.87483	4.03920	8.70219	.151745
6.60	43.5600	2.56905	8.12404	287.496	1.87578	4.04124	8.70659	.151515
6.61	43.6921	2.57099	8.13019	288.805	1.87672	4.04328	8.71098	.151286
6.62	43.8244	2.57294	8.13634	290.118	1.87767	4.04532	8.71537	.151057
6.63	43.9569	2.57488	8.14248	291.434	1.87862	4.04735	8.71976	.150830
6.64	44.0896	2.57682	8.14862	292.755	1.87956	4.04939	8.72414	.150602
6.65	44.2225	2.57876	8.15475	294.080	1.88050	4.05142	8.72852	.150376
6.66	44.3556	2.58070	8.16088	295.408	1.88144	4.05345	8.73289	.150150
6.67	44.4889	2.58263	8.16701	296.741	1.88239	4.05548	8.73726	.149925
6.68	44.6224	2.58457	8.17313	298.078	1.88333	4.05750	8.74162	.149701
6.69	44.7561	2.58650	8.17924	299.418	1.88427	4.05953	8.74598	.149477
6.70	44.8900	2.58844	8.18535	300.763	1.88520	4.06155	8.75034	.149254
6.71	45.0241	2.59037	8.19146	302.112	1.88614	4.06357	8.75469	.149031
6.72	45.1584	2.59230	8.19756	303.464	1.88708	4.06559	8.75904	.148810
6.73	45.2929	2.59422	8.20366	304.821	1.88801	4.06760	8.76338	.148588
6.74	45.4276	2.59615	8.20975	306.182	1.88895	4.06961	8.76772	.148368
6.75	45.5625	2.59808	8.21584	307.547	1.88988	4.07163	8.77205	.148148
6.76	45.6976	2.60000	8.22192	308.916	1.89081	4.07364	8.77638	.147929
6.77	45.8329	2.60192	8.22800	310.289	1.89175	4.07564	8.78071	.147710
6.78	45.9684	2.60384	8.23408	311.666	1.89268	4.07765	8.78503	.147493
6.79	46.1041	2.60576	8.24015	313.047	1.89361	4.07965	8.78935	.147275
6.80	46.2400	2.60768	8.24621	314.432	1.89454	4.08166	8.79366	.147059
6.81	46.3761	2.60960	8.25227	315.821	1.89546	4.08365	8.79797	.146843
6.82	46.5124	2.61151	8.25833	317.215	1.89639	4.08565	8.80227	.146628
6.83	46.6489	2.61343	8.26438	318.612	1.89732	4.08765	8.80657	.146413
6.84	46.7856	2.61534	8.27043	320.014	1.89824	4.08964	8.81087	.146199
6.85	46.9225	2.61725	8.27647	321.419	1.89917	4.09163	8.81516	.145985
6.86	47.0596	2.61916	8.28251	322.829	1.90009	4.09362	8.81945	.145773
6.87	47.1969	2.62107	8.28855	324.243	1.90102	4.09561	8.82373	.145560
6.88	47.3344	2.62298	8.29458	325.661	1.90194	4.09760	8.82801	.145349
6.89	47.4721	2.62488	8.30060	327 083	1.90286	4.09958	8.83228	.145138
6.90	47.6100	2.62679	8.30662	328.509	1.90378	4.10157	8 83656	.144928
6.91	47.7481	2.62869	8.31264	329.939	1.90470	4.10355	8.84082	.144718
6.92	47.8864	2.63059	8.31865	331.374	1.90562	4.10552	8.84509	.144509
6.93	48.0249	2.63249	8.32466	332.813	1.90653	4.10750	8.84934	.144300
6.94	48.1636	2.63439	8.33067	334.255	1.90745	4.10948	8.85360	.144092
6.95	48.3025	2.63629	8.33667	335.702	1.90837	4.11145	8.85785	.143885
6.96	48.4416	2.63818	8.34266	337.154	1.90928	4.11342	8.86210	.143678
6.97	48.5809	2.64008	8.34865	338.609	1.91019	4.11539	8.86634	.143472
6.98	48.7204	2.64197	8.35464	340.068	1.91111	4.11736	8.87058	.143266
6.99	48.8601	2.64386	8.36062	341.532	1.91202	4.11932	8.87481	.143062
7.00	49.0000	2.64575	8.36660	343.000	1.91293	4.12129	8.87904	.142857
n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$

n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$
7.00	49.0000	2.64575	8.36660	343.000	1.91293	4.12129	8.87904	.142857
7.01	49.1401	2.64764	8.37257	344.472	1.91384	4.12325	8.88327	.142653
7.02	49.2804	2.64953	8.37854	345.948	1.91475	4.12521	8.88749	.142450
7.03	49.4209	2.65141	8.38451	347.429	1.91566	4.12716	8.89171	.142248
7.04	49.5616	2.65330	8.39047	348.914	1.91657	4.12912	8.89592	.142045
7.05	49.7025	2.65518	8.39643	350.403	1.91747	4.13107	8.90013	.141844
7.06	49.8436	2.65707	8.40238	351.896	1.91838	4.13303	8.90434	.141643
7.07	49.9849	2.65895	8.40833	353.393	1.91929	4.13498	8.90854	.141443
7.08	50.1264	2.66083	8.41427	354.895	1.92019	4.13693	8.91274	.141243
7.09	50.2681	2.66271	8.42021	356.401	1.92109	4.13887	8.91693	.141044
7.10	50.4100	2.66458	8.42615	357.911	1.92200	4.14082	8.92112	.140845
7.11	50.5521	2.66646	8.43208	359.425	1.92290	4.14276	8.92531	.140647
7.12	50.6944	2.66833	8.43801	360.944	1.92380	4.14470	8.92949	.140449
7.13	50.8369	2.67021	8.44393	362.467	1.92470	4.14664	8.93367	.140252
7.14	50.9796	2.67208	8.44985	363.994	1.92560	4.14858	8.93784	.140056
7.15	51.1225	2.67395	8.45577	365.526	1.92650	4.15052	8.94201	.139860
7.16	51.2656	2.67582	8.46168	367.062	1.92740	4.15245	8.94618	.139665
7.17	51.4089	2.67769	8.46759	368.602	1.92829	4.15438	8.95034	.139470
7.18	51.5524	2.67955	8.47349	370.146	1.92919	4.15631	8.95450	.139276
7.19	51.6961	2.68142	8.47939	371.695	1.93008	4.15824	8.95866	.139082
7.20	51.8400	2.68328	8.48528	373.248	1.93098	4.16017	8.96281	.138889
7.21	51.9841	2.68514	8.49117	374.805	1.93187	4.16209	8.96696	.138696
7.22	52.1284	2.68701	8.49706	376.367	1.93277	4.16402	8.97110	.138504
7.23	52.2729	2.68887	8.50294	377.933	1.93366	4.16594	8.97524	.138313
7.24	52.4176	2.69072	8.50882	379.503	1.93455	4.16786	8.97938	.138122
7.25	52.5625	2.69258	8.51469	381.078	1.93544	4.16978	8.98351	.137931
7.26	52.7076	2.69444	8.52056	382.657	1.93633	4.17169	8.98764	.137741
7.27	52.8529	2.69629	8.52643	384.241	1.93722	4.17361	8.99176	.137552
7.28	52.9984	2.69815	8.53229	385.828	1.93810	4.17552	8.99588	.137363
7.29	53.1441	2.70000	8.53815	387.420	1.93899	4.17743	9.00000	.137174
7.30	53.2900	2.70185	8.54400	389.017	1.93988	4.17934	9.00411	.136986
7.31	53.4361	2.70370	8.54985	390.618	1.94076	4.18125	9.00822	.136799
7.32	53.5824	2.70555	8.55570	392.223	1.94165	4.18315	9.01233	.136612
7.33	53.7289	2.70740	8.56154	393.833	1.94253	4.18506	9.01643	.136426
7.34	53.8756	2.70924	8.56738	395.447	1.94341	4.18696	9.02053	.136240
7.35	54.0225	2.71109	8.57321	397.065	1.94430	4.18886	9.02462	.136054
7.36	54.1696	2.71293	8.57904	398.688	1.94518	4.19076	9.02871	.135870
7.37	54.3169	2.71477	8.58487	400.316	1.94606	4.19266	9.03280	.135685
7.38	54.4644	2.71662	8.59069	401.947	1.94694	4.19455	9.03689	.135501
7.39	54.6121	2.71846	8.59651	403.583	1.94782	4.19644	9.04097	.135318
7.40	54.7600	2.72029	8.60233	405.224	1.94870	4.19834	9.04504	.135135
7.41	54.9081	2.72213	8.60814	406.869	1.94957	4.20023	9.04911	.134953
7.42	55.0564	2.72397	8.61394	408.518	1.95045	4.20212	9.05318	.134771
7.43	55.2049	2.72580	8.61974	410.172	1.95132	4.20400	9.05725	.134590
7.44	55.3536	2.72764	8.62554	411.831	1.95220	4.20589	9.06131	.134409
7.45	55.5025	2.72947	8.63134	413.494	1.95307	4.20777	9.06537	.134228
7.46	55.6516	2.73130	8.63713	415.161	1.95395	4.20965	9.06942	.134048
7.47	55.8009	2.73313	8.64292	416.833	1.95482	4.21153	9.07347	.133869
7.48	55.9504	2.73496	8.64870	418.509	1.95569	4.21341	9.07752	.133690
7.49	56.1001	2.73679	8.65448	420.190	1.95656	4.21529	9.08156	.133511
7.50	56.2500	2.73861	8.66025	421.875	1.95743	4.21716	9.08560	.133333
n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$

n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$
7.50	56.2500	2.73861	8.66025	421.875	1.95743	4.21716	9.08560	.133333
7.51	56.4001	2.74044	8.66603	423.565	1.95830	4.21904	9.08964	.133156
7.52	56.5504	2.74226	8.67179	425.259	1.95917	4.22091	9.09367	.132979
7.53	56.7009	2.74408	8.67756	426.958	1.96004	4.22278	9.09770	.132802
7.54	56.8516	2.74591	8.68332	428.661	1.96091	4.22465	9.10173	.132626
7.55	57.0025	2.74773	8.68907	430.369	1.96177	4.22651	9.10575	.132450
7.56	57.1536	2.74955	8.69483	432.081	1.96264	4.22838	9.10977	.132275
7.57	57.3049	2.75136	8.70057	433.798	1.96350	4.23024	9.11378	.132100
7.58	57.4564	2.75318	8.70632	435.520	1.96437	4.23210	9.11779	.131926
7.59	57.6081	2.75500	8.71206	437.245	1.96523	4.23396	9.12180	.131752
7.60	57.7600	2.75681	8.71780	438.976	1.96610	4.23582	9.12581	.131579
7.61	57.9121	2.75862	8.72353	440.711	1.96696	4.23768	9.12981	.131406
7.62	58.0644	2.76043	8.72926	442.451	1.96782	4.23954	9.13380	.131234
7.63	58.2169	2.76225	8.73499	444.195	1.96868	4.24139	9.13780	.131062
7.64	58.3696	2.76405	8.74071	445.944	1.96954	4.24324	9.14179	.130890
7.65	58.5225	2.76586	8.74643	447.697	1.97040	4.24509	9.14577	.130719
7.66	58.6756	2.76767	8.75214	449.455	1.97126	4.24694	9.14976	.130548
7.67	58.8289	2.76948	8.75785	451.218	1.97211	4.24879	9.15374	.130378
7.68	58.9824	2.77128	8.76356	452.985	1.97297	4.25063	9.15771	.130208
7.69	59.1361	2.77308	8.76926	454.757	1.97383	4.25248	9.16169	.130039
7.70	59.2900	2.77489	8.77496	456.533	1.97468	4.25432	9.16566	.129870
7.71	59.4441	2.77669	8.78066	458.314	1.97554	4.25616	9.16962	.129702
7.72	59.5984	2.77849	8.78635	460.100	1.97639	4.25800	9.17359	.129534
7.73	59.7529	2.78029	8.79204	461.890	1.97724	4.25984	9.17754	.129366
7.74	59.9076	2.78209	8.79773	463.685	1.97809	4.26167	9.18150	.129199
7.75	60.0625	2.78388	8.80341	465.484	1.97895	4.26351	9.18545	.129032
7.76	60.2176	2.78568	8.80909	467.289	1.97980	4.26534	9.18940	.128866
7.77	60.3729	2.78747	8.81476	469.097	1.98065	4.26717	9.19335	.128700
7.78	60.5284	2.78927	8.82043	470.911	1.98150	4.26900	9.19729	.128535
7.79	60.6841	2.79106	8.82610	472.729	1.98234	4.27083	9.20123	.128370
7.80	60.8400	2.79285	8.83176	474.552	1.98319	4.27266	9.20516	.128205
7.81	60.9961	2.79464	8.83742	476.380	1.98404	4.27448	9.20910	.128041
7.82	61.1524	2.79643	8.84308	478.212	1.98489	4.27631	9.21302	.127877
7.83	61.3089	2.79821	8.84873	480.049	1.98573	4.27813	9.21695	.127714
7.84	61.4656	2.80000	8.85438	481.890	1.98658	4.27995	9.22087	.127551
7.85	61.6225	2.80179	8.86002	483.737	1.98742	4.28177	9.22479	.127389
7.86	61.7796	2.80357	8.86566	485.588	1.98826	4.28359	9.22871	.127226
7.87	61.9369	2.80535	8.87130	487.443	1.98911	4.28540	9.23262	.127065
7.88	62.0944	2.80713	8.87694	489.304	1.98995	4.28722	9.23653	.126904
7.89	62.2521	2.80891	8.88257	491.169	1.99079	4.28903	9.24043	.126743
7.90	62.4100	2.81069	8.88819	493.039	1.99163	4.29084	9.24434	.126582
7.91	62.5681	2.81247	8.89382	494.914	1.99247	4.29265	9.24823	.126422
7.92	62.7264	2.81425	8.89944	496.793	1.99331	4.29446	9.25213	.126263
7.93	62.8849	2.81603	8.90505	498.677	1.99415	4.29627	9.25602	.126103
7.94	63.0436	2.81780	8.91067	500.566	1.99499	4.29807	9.25991	.125945
7.95	63.2025	2.81957	8.91628	502.460	1.99582	4.29987	9.26380	.125786
7.96	63.3616	2.82135	8.92188	504.358	1.99666	4.30168	9.26768	.125628
7.97	63.5209	2.82312	8.92749	506.262	1.99750	4.30348	9.27156	.125471
7.98	63.6804	2.82489	8.93308	508.170	1.99833	4.30528	9.27544	.125313
7.99	63.8401	2.82666	8.93868	510.082	1.99917	4.30707	9.27931	.125156
8.00	64.0000	2.82843	8.94427	512.000	2.00000	4.30887	9.28318	.125000
n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$

n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$
8.00	64.0000	2.82843	8.94427	512.000	2.00000	4.30887	9.28318	.125000
8.01	64.1601	2.83019	8.94986	513.922	2.00083	4.31066	9.28704	.124844
8.02	64.3204	2.83196	8.95545	515.850	2.00167	4.31246	9.29091	.124688
8.03	64.4809	2.83373	8.96103	517.782	2.00250	4.31425	9.29477	.124533
8.04	64.6416	2.83549	8.96660	519.718	2.00333	4.31604	9.29862	.124378
8.05	64.8025	2.83725	8.97218	521.660	2.00416	4.31783	9.30248	.124224
8.06	64.9636	2.83901	8.97775	523.607	2.00499	4.31961	9.30633	.124069
8.07	65.1249	2.84077	8.98332	525.558	2.00582	4.32140	9.31018	.123916
8.08	65.2864	2.84253	8.98888	527.514	2.00664	4.32318	9.31402	.123762
8.09	65.4481	2.84429	8.99444	529.475	2.00747	4.32497	9.31786	.123609
8.10	65.6100	2.84605	9.00000	531.441	2.00830	4.32675	9.32170	.123457
8.11	65.7721	2.84781	9.00555	533.412	2.00912	4.32853	9.32553	.123305
8.12	65.9344	2.84956	9.01110	535.387	2.00995	4.33031	9.32936	.123153
8.13	66.0969	2.85132	9.01665	537.368	2.01078	4.33208	9.33319	.123001
8.14	66.2596	2.85307	9.02219	539.353	2.01160	4.33386	9.33702	.122850
8.15	66.4225	2.85482	9.02774	541.343	2.01242	4.33563	9.34084	.122699
8.16	66.5856	2.85657	9.03327	543.338	2.01325	4.33741	9.34466	.122549
8.17	66.7489	2.85832	9.03881	545.339	2.01407	4.33918	9.34847	.122399
8.18	66.9124	2.86007	9.04434	547.343	2.01489	4.34095	9.35229	.122249
8.19	67.0761	2.86182	9.04986	549.353	2.01571	4.34271	9.35610	.122100
8.20	67.2400	2.86356	9.05539	551.368	2.01653	4.34448	9.35990	.121951
8.21	67.4041	2.86531	9.06091	553.388	2.01735	4.34625	9.36370	.121803
8.22	67.5684	2.86705	9.06642	555.412	2.01817	4.34801	9.36751	.121655
8.23	67.7329	2.86880	9.07193	557.442	2.01899	4.34977	9.37130	.121507
8.24	67.8976	2.87054	9.07744	559.476	2.01980	4.35153	9.37510	.121359
8.25	68.0625	2.87228	9.08295	561.516	2.02062	4.35329	9.37889	.121212
8.26	68.2276	2.87402	9.08845	563.560	2.02144	4.35505	9.38268	.121065
8.27	68.3929	2.87576	9.09395	565.609	2.02225	4.35681	9.38646	.120919
8.28	68.5584	2.87750	9.09945	567.664	2.02307	4.35856	9.39024	.120773
8.29	68.7241	2.87924	9.10494	569.723	2.02388	4.36032	9.39402	.120627
8.30	68.8900	2.88097	9.11043	571.787	2.02469	4.36207	9.39780	.120482
8.31	69.0561	2.88271	9.11592	573.856	2.02551	4.36382	9.40157	.120337
8.32	69.2224	2.88444	9.12140	575.930	2.02632	4.36557	9.40534	.120192
8.33	69.3889	2.88617	9.12688	578.010	2.02713	4.36732	9.40911	.120048
8.34	69.5556	2.88791	9.13236	580.094	2.02794	4.36907	9.41287	.119904
8.35	69.7225	2.88964	9.13783	582.183	2.02875	4.37081	9.41663	.119760
8.36	69.8896	2.89137	9.14330	584.277	2.02956	4.37256	9.42039	.119617
8.37	70.0569	2.89310	9.14877	586.376	2.03037	4.37430	9.42414	.119474
8.38	70.2244	2.89482	9.15423	588.480	2.03118	4.37604	9.42789	.119332
8.39	70.3921	2.89655	9.15969	590.590	2.03199	4.37778	9.43164	.119190
8.40	70.5600	2.89828	9.16515	592.704	2.03279	4.37952	9.43539	.119048
8.41	70.7281	2.90000	9.17061	594.823	2.03360	4.38126	9.43913	.118906
8.42	70.8964	2.90172	9.17606	596.948	2.03440	4.38299	9.44287	.118765
8.43	71.0649	2.90345	9.18150	599.077	2.03521	4.38473	9.44661	.118624
8.44	71.2336	2.90517	9.18695	601.212	2.03601	4.38646	9.45034	.118483
8.45	71.4025	2.90689	9.19239	603.351	2.03682	4.38819	9.45407	.118343
8.46	71.5716	2.90861	9.19783	605.496	2.03762	4.38992	9.45780	.118203
8.47	71.7409	2.91033	9.20326	607.645	2.03842	4.39165	9.46152	.118064
8.48	71.9104	2.91204	9.20869	609.800	2.03923	4.39338	9.46525	.117925
8.49	72.0801	2.91376	9.21412	611.960	2.04003	4.39510	9.46897	.117786
8.50	72.2500	2.91548	9.21954	614.125	2.04083	4.39683	9.47268	.117647
n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$

n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$
8.50	72.2500	2.91548	9.21954	614.125	2.04083	4.39683	9.47268	.117647
8.51	72.4201	2.91719	9.22497	616.295	2.04163	4.39855	9.47640	.117509
8.52	72.5904	2.91890	9.23038	618.470	2.04243	4.40028	9.48011	.117371
8.53	72.7609	2.92062	9.23580	620.650	2.04325	4.40200	9.48381	.117233
8.54	72.9316	2.92233	9.24121	622.836	2.04402	4.40372	9.48752	.117096
8.55	73.1025	2.92404	9.24662	625.026	2.04482	4.40543	9.49122	.116959
8.56	73.2736	2.92575	9.25203	627.222	2.04562	4.40715	9.49492	.116822
8.57	73.4449	2.92746	9.25743	629.423	2.04641	4.40887	9.49861	.116686
8.58	73.6164	2.92916	9.26283	631.629	2.04721	4.41058	9.50231	.116550
8.59	73.7881	2.93087	9.26823	633.840	2.04801	4.41229	9.50600	.116414
8.60	73.9600	2.93258	9.27362	636.056	2.04880	4.41400	9.50969	.116279
8.61	74.1321	2.93428	9.27901	638.277	2.04959	4.41571	9.51337	.116144
8.62	74.3044	2.93598	9.28440	640.504	2.05039	4.41742	9.51705	.116009
8.63	74.4769	2.93769	9.28978	642.736	2.05118	4.41913	9.52073	.115875
8.64	74.6496	2.93939	9.29516	644.973	2.05197	4.42084	9.52441	.115741
8.65	74.8225	2.94109	9.30054	647.215	2.05276	4.42254	9.52808	.115607
8.66	74.9956	2.94279	9.30591	649.462	2.05355	4.42425	9.53175	.115473
8.67	75.1689	2.94449	9.31128	651.714	2.05434	4.42595	9.53542	.115340
8.68	75.3424	2.94618	9.31665	653.972	2.05513	4.42765	9.53908	.115207
8.69	75.5161	2.94788	9.32202	656.235	2.05592	4.42935	9.54274	.115075
8.70	75.6900	2.94958	9.32738	658.503	2.05671	4.43105	9.54640	.114943
8.71	75.8641	2.95127	9.33274	660.776	2.05750	4.43274	9.55006	.114811
8.72	76.0384	2.95296	9.33809	663.055	2.05828	4.43444	9.55371	.114679
8.73	76.2129	2.95466	9.34345	665.339	2.05907	4.43613	9.55736	.114548
8.74	76.3876	2.95635	9.34880	667.628	2.05986	4.43783	9.56101	.114416
8.75	76.5625	2.95804	9.35414	669.922	2.06064	4.43952	9.56466	.114286
8.76	76.7376	2.95973	9.35949	672.221	2.06143	4.44121	9.56830	.114155
8.77	76.9129	2.96142	9.36483	674.526	2.06221	4.44290	9.57194	.114025
8.78	77.0884	2.96311	9.37017	676.836	2.06299	4.44459	9.57557	.113895
8.79	77.2641	2.96479	9.37550	679.151	2.06378	4.44627	9.57921	.113766
8.80	77.4400	2.96648	9.38083	681.472	2.06456	4.44796	9.58284	.113636
8.81	77.6161	2.96816	9.38616	683.798	2.06534	4.44964	9.58647	.113507
8.82	77.7924	2.96985	9.39149	686.129	2.06612	4.45133	9.59009	.113379
8.83	77.9689	2.97153	9.39681	688.465	2.06690	4.45301	9.59372	.113250
8.84	78.1456	2.97321	9.40213	690.807	2.06768	4.45469	9.59734	.113122
8.85	78.3225	2.97489	9.40744	693.154	2.06846	4.45637	9.60095	.112994
8.86	78.4996	2.97658	9.41276	695.506	2.06924	4.45805	9.60457	.112867
8.87	78.6769	2.97825	9.41807	697.864	2.07002	4.45972	9.60818	.112740
8.88	78.8544	2.97993	9.42338	700.227	2.07080	4.46140	9.61179	.112613
8.89	79.0321	2.98161	9.42868	702.595	2.07157	4.46307	9.61540	.112486
8.90	79.2100	2.98329	9.43398	704.969	2.07235	4.46475	9.61900	.112360
8.91	79.3881	2.98496	9.43928	707.348	2.07313	4.46642	9.62260	.112233
8.92	79.5664	2.98664	9.44458	709.732	2.07390	4.46809	9.62620	.112108
8.93	79.7449	2.98831	9.44987	712.122	2.07468	4.46976	9.62980	.111982
8.94	79.9236	2.98998	9.45516	714.517	2.07545	4.47142	9.63339	.111857
8.95	80.1025	2.99166	9.46044	716.917	2.07622	4.47309	9.63698	.111732
8.96	80.2816	2.99333	9.46573	719.323	2.07700	4.47476	9.64057	.111607
8.97	80.4609	2.99500	9.47101	721.734	2.07777	4.47642	9.64415	.111483
8.98	80.6404	2.99666	9.47629	724.151	2.07854	4.47808	9.64774	.111359
8.99	80.8201	2.99833	9.48156	726.573	2.07931	4.47974	9.65132	.111235
9.00	81.0000	3.00000	9.48683	729.000	2.08008	4.48140	9.65489	.111111
n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$

n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$
9.00	81.0000	3.00000	9.48683	729.000	2.08008	4.48140	9.65489	.111111
9.01	81.1801	3.00167	9.49210	731.433	2.08085	4.48306	9.65847	.110988
9.02	81.3604	3.00333	9.49737	733.871	2.08162	4.48472	9.66204	.110865
9.03	81.5409	3.00500	9.50263	736.314	2.08239	4.48638	9.66561	.110742
9.04	81.7216	3.00666	9.50789	738.763	2.08316	4.48803	9.66918	.110619
9.05	81.9025	3.00832	9.51315	741.218	2.08393	4.48969	9.67274	.110497
9.06	82.0836	3.00998	9.51840	743.677	2.08470	4.49134	9.67630	.110375
9.07	82.2649	3.01164	9.52365	746.143	2.08546	4.49299	9.67986	.110254
9.08	82.4464	3.01330	9.52890	748.613	2.08623	4.49464	9.68342	.110132
9.09	82.6281	3.01496	9.53415	751.089	2.08699	4.49629	9.68697	.110011
9.10	82.8100	3.01662	9.53939	753.571	2.08776	4.49794	9.69052	.109890
9.11	82.9921	3.01828	9.54463	756.058	2.08852	4.49959	9.69407	.109769
9.12	83.1744	3.01993	9.54987	758.551	2.08929	4.50123	9.69762	.109649
9.13	83.3569	3.02159	9.55510	761.048	2.09005	4.50288	9.70116	.109529
9.14	83.5396	3.02324	9.56033	763.552	2.09081	4.50452	9.70470	.109409
9.15	83.7225	3.02490	9.56556	766.061	2.09158	4.50616	9.70824	.109290
9.16	83.9056	3.02655	9.57079	768.575	2.09234	4.50781	9.71177	.109170
9.17	84.0889	3.02820	9.57601	771.095	2.09310	4.50945	9.71531	.109051
9.18	84.2724	3.02985	9.58123	773.621	2.09386	4.51108	9.71884	.108932
9.19	84.4561	3.03150	9.58645	776.152	2.09462	4.51272	9.72236	.108814
9.20	84.6400	3.03315	9.59166	778.688	2.09538	4.51436	9.72589	.108696
9.21	84.8241	3.03480	9.59687	781.230	2.09614	4.51599	9.72941	.108578
9.22	85.0084	3.03645	9.60208	783.777	2.09690	4.51763	9.73293	.108460
9.23	85.1929	3.03809	9.60729	786.330	2.09765	4.51926	9.73645	.108342
9.24	85.3776	3.03974	9.61249	788.889	2.09841	4.52089	9.73996	.108225
9.25	85.5625	3.04138	9.61769	791.453	2.09917	4.52252	9.74348	.108108
9.26	85.7476	3.04302	9.62289	794.023	2.09992	4.52415	9.74699	.107991
9.27	85.9329	3.04467	9.62808	796.598	2.10068	4.52578	9.75049	.107875
9.28	86.1184	3.04631	9.63328	799.179	2.10144	4.52740	9.75400	.107759
9.29	86.3041	3.04795	9.63846	801.765	2.10219	4.52903	9.75750	.107643
9.30	86.4900	3.04959	9.64365	804.357	2.10294	4.53065	9.76100	.107527
9.31	86.6761	3.05123	9.64883	806.954	2.10370	4.53228	9.76450	.107411
9.32	86.8624	3.05287	9.65401	809.558	2.10445	4.53390	9.76799	.107296
9.33	87.0489	3.05450	9.65919	812.166	2.10520	4.53552	9.77148	.107181
9.34	87.2356	3.05614	9.66437	814.781	2.10595	4.53714	9.77497	.107066
9.35	87.4225	3.05778	9.66954	817.400	2.10671	4.53876	9.77846	.106952
9.36	87.6096	3.05941	9.67471	820.026	2.10746	4.54038	9.78195	.106838
9.37	87.7969	3.06105	9.67988	822.657	2.10821	4.54199	9.78543	.106724
9.38	87.9844	3.06268	9.68504	825.294	2.10896	4.54361	9.78891	.106610
9.39	88.1721	3.06431	9.69020	827.936	2.10971	4.54522	9.79239	.106496
9.40	88.3600	3.06594	9.69536	830.584	2.11045	4.54684	9.79586	.106383
9.41	88.5481	3.06757	9.70052	833.238	2.11120	4.54845	9.79933	.106270
9.42	88.7364	3.06920	9.70567	835.897	2.11195	4.55006	9.80280	.106157
9.43	88.9249	3.07083	9.71082	838.562	2.11270	4.55167	9.80627	.106045
9.44	89.1136	3.07246	9.71597	841.232	2.11344	4.55328	9.80974	.105932
9.45	89.3025	3.07409	9.72111	843.909	2.11419	4.55488	9.81320	.105820
9.46	89.4916	3.07571	9.72625	846.591	2.11494	4.55649	9.81666	.105708
9.47	89.6809	3.07734	9.73139	849.278	2.11568	4.55809	9.82012	.105597
9.48	89.8704	3.07896	9.73653	851.971	2.11642	4.55970	9.82357	.105485
9.49	90.0601	3.08058	9.74166	854.670	2.11717	4.56130	9.82703	.105374
9.50	90.2500	3.08221	9.74679	857.375	2.11791	4.56290	9.83048	.105263
n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$

n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$
9.50	90.2500	3.08221	9.74679	857.375	2.11791	4.56290	9.83048	.105263
9.51	90.4401	3.08383	9.75192	860.085	2.11865	4.56450	9.83392	.105152
9.52	90.6304	3.08545	9.75705	862.801	2.11940	4.56610	9.83737	.105042
9.53	90.8209	3.08707	9.76217	865.523	2.12014	4.56770	9.84081	.104932
9.54	91.0116	3.08869	9.76729	868.251	2.12088	4.56930	9.84425	.104822
9.55	91.2025	3.09031	9.77241	870.984	2.12162	4.57089	9.84769	.104712
9.56	91.3936	3.09192	9.77753	873.723	2.12236	4.57249	9.85113	.104603
9.57	91.5849	3.09354	9.78264	876.467	2.12310	4.57408	9.85456	.104493
9.58	91.7764	3.09516	9.78775	879.218	2.12384	4.57567	9.85799	.104384
9.59	91.9681	3.09677	9.79285	881.974	2.12458	4.57727	9.86142	.104275
9.60	92.1600	3.09839	9.79796	884.736	2.12532	4.57886	9.86485	.104167
9.61	92.3521	3.10000	9.80306	887.504	2.12605	4.58045	9.86827	.104058
9.62	92.5444	3.10161	9.80816	890.277	2.12679	4.58204	9.87169	.103950
9.63	92.7369	3.10322	9.81326	893.056	2.12753	4.58362	9.87511	.103842
9.64	92.9296	3.10483	9.81835	895.841	2.12826	4.58521	9.87853	.103734
9.65	93.1225	3.10644	9.82344	898.632	2.12900	4.58679	9.88195	.103627
9.66	93.3156	3.10805	9.82853	901.429	2.12974	4.58838	9.88536	.103520
9.67	93.5089	3.10966	9.83362	904.231	2.13047	4.58996	9.88877	.103413
9.68	93.7024	3.11127	9.83870	907.039	2.13120	4.59154	9.89217	.103306
9.69	93.8961	3.11288	9.84378	909.853	2.13194	4.59312	9.89558	.103199
9.70	94.0900	3.11448	9.84886	912.673	2.13267	4.59470	9.89898	.103093
9.71	94.2841	3.11609	9.85393	915.499	2.13340	4.59628	9.90238	.102987
9.72	94.4784	3.11769	9.85901	918.330	2.13414	4.59786	9.90578	.102881
9.73	94.6729	3.11929	9.86408	921.167	2.13487	4.59943	9.90918	.102775
9.74	94.8676	3.12090	9.86914	924.010	2.13560	4.60101	9.91257	.102669
9.75	95.0625	3.12250	9.87421	926.859	2.13633	4.60258	9.91596	.102564
9.76	95.2576	3.12410	9.87927	929.714	2.13706	4.60416	9.91935	.102459
9.77	95.4529	3.12570	9.88433	932.575	2.13779	4.60573	9.92274	.102354
9.78	95.6484	3.12730	9.88939	935.441	2.13852	4.60730	9.92612	.102249
9.79	95.8441	3.12890	9.89444	938.314	2.13925	4.60887	9.92950	.102145
9.80	96.0400	3.13050	9.89949	941.192	2.13997	4.61044	9.93288	.102041
9.81	96.2361	3.13209	9.90454	944.076	2.14070	4.61200	9.93626	.101937
9.82	96.4324	3.13369	9.90959	946.966	2.14143	4.61357	9.93964	.101833
9.83	96.6289	3.13528	9.91464	949.862	2.14216	4.61514	9.94301	.101729
9.84	96.8256	3.13688	9.91968	952.764	2.14288	4.61670	9.94638	.101626
9.85	97.0225	3.13847	9.92472	955.672	2.14361	4.61826	9.94975	.101523
9.86	97.2196	3.14006	9.92975	958.585	2.14433	4.61983	9.95311	.101420
9.87	97.4169	3.14166	9.93479	961.505	2.14506	4.62139	9.95648	.101317
9.88	97.6144	3.14325	9.93982	964.430	2.14578	4.62295	9.95984	.101215
9.89	97.8121	3.14484	9.94485	967.362	2.14651	4.62451	9.96320	.101112
9.90	98.0100	3.14643	9.94987	970.299	2.14723	4.62607	9.96655	.101010
9.91	98.2081	3.14802	9.95490	973.242	2.14795	4.62762	9.96991	.100908
9.92	98.4064	3.14960	9.95992	976.191	2.14867	4.62918	9.97326	.100806
9.93	98.6049	3.15119	9.96494	979.147	2.14940	4.63073	9.97661	.100705
9.94	98.8036	3.15278	9.96995	982.108	2.15012	4.63229	9.97996	.100604
9.95	99.0025	3.15436	9.97497	985.075	2.15084	4.63384	9.98331	.100503
9.96	99.2016	3.15595	9.97998	988.048	2.15156	4.63539	9.98665	.100402
9.97	99.4009	3.15753	9.98499	991.027	2.15228	4.63694	9.98999	.100301
9.98	99.6004	3.15911	9.98999	994.012	2.15300	4.63849	9.99333	.100200
9.99	99.8001	3.16070	9.99500	997.003	2.15372	4.64004	9.99667	.100100
10.00	100.000	3.16228	10.0000	1000.00	2.15443	4.64159	10.0000	.100000
n	n^2	$\sqrt{n}$	$\sqrt{10\,n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10\,n}$	$\sqrt[3]{100\,n}$	$1/n$

Table III*a*—Compound Interest: $(1 + r)^n$

Amount of One Dollar Principal at Compound Interest After n Years

n	2 %	2½%	3 %	3½%	4 %	4½%	5 %	6 %	7 %
1	1.0200	1.0250	1.0300	1.0350	1.0400	1.0450	1.0500	1.0600	1.0700
2	1.0404	1.0506	1.0609	1.0712	1.0816	1.0920	1.1025	1.1236	1.1449
3	1.0612	1.0769	1.0927	1.1087	1.1249	1.1412	1.1576	1.1910	1.2250
4	1.0824	1.1038	1.1255	1.1475	1.1699	1.1925	1.2155	1.2625	1.3108
5	1.1041	1.1314	1.1593	1.1877	1.2167	1.2462	1.2763	1.3382	1.4026
6	1.1262	1.1597	1.1941	1.2293	1.2653	1.3023	1.3401	1.4185	1.5007
7	1.1487	1.1887	1.2299	1.2723	1.3159	1.3609	1.4071	1.5036	1.6058
8	1.1717	1.2184	1.2668	1.3168	1.3686	1.4221	1.4775	1.5938	1.7182
9	1.1951	1.2489	1.3048	1.3629	1.4233	1.4861	1.5513	1.6895	1 8385
10	1.2190	1.2801	1.3439	1.4106	1.4802	1.5530	1.6289	1.7908	1 9672
11	1.2434	1.3121	1.3842	1.4600	1.5395	1.6229	1.7103	1.8983	2.1049
12	1.2682	1.3449	1.4258	1.5111	1.6010	1.6959	1.7959	2.0122	2.2522
13	1.2936	1.3785	1.4685	1.5640	1.6651	1.7722	1.8856	2.1329	2.4098
14	1.3195	1.4130	1.5126	1.6187	1.7317	1.8519	1.9799	2.2609	2.5785
15	1.3459	1.4483	1.5580	1.6753	1.8009	1.9353	2.0789	2.3966	2.7590
16	1.3728	1.4845	1.6047	1.7340	1.8730	2.0224	2.1829	2.5404	2.9522
17	1.4002	1.5216	1.6528	1.7947	1.9479	2.1134	2.2920	2.6928	3.1588
18	1.4282	1.5597	1.7024	1.8575	2.0258	2.2085	2.4066	2.8543	3.3799
19	1.4568	1.5987	1.7535	1.9225	2.1068	2.3079	2.5270	3.0256	3 6165
20	1.4859	1.6386	1.8061	1.9898	2 1911	2.4117	2.6533	3.2071	3.8697
21	1.5157	1.6796	1.8603	2.0594	2.2788	2.5202	2.7860	3.3996	4.1406
22	1.5460	1.7216	1.9161	2.1315	2.3699	2.6337	2.9253	3.6035	4.4304
23	1.5769	1.7646	1.9736	2.2061	2.4647	2.7522	3.0715	3.8197	4.7405
24	1.6084	1.8087	2.0328	2.2833	2.5633	2.8760	3.2251	4.0489	5.0724
25	1.6406	1.8539	2.0938	2.3632	2.6658	3.0054	3.3864	4.2919	5.4274
26	1.6734	1.9003	2.1566	2.4460	2.7725	3.1407	3.5557	4.5494	5.8074
27	1.7069	1.9478	2.2213	2.5316	2.8834	3.2820	3.7335	4.8223	6.2139
28	1.7410	1.9965	2.2879	2.6202	2.9987	3.4297	3.9201	5.1117	6.6488
29	1.7758	2.0464	2.3566	2.7119	3.1187	3.5840	4.1161	5.4184	7.1143
30	1.8114	2.0976	2.4273	2.8068	3.2434	3.7453	4.3219	5.7435	7.6123
31	1.8476	2.1500	2.5001	2.9050	3.3731	3.9139	4.5380	6.0881	8.1451
32	1.8845	2.2038	2.5751	3.0067	3.5081	4.0900	4.7649	6.4534	8.7153
33	1.9222	2.2589	2.6523	3.1119	3.6484	4.2740	5.0032	6.8406	9.3253
34	1.9607	2.3153	2.7319	3.2209	3.7943	4.4664	5.2533	7.2510	9.9781
35	1.9999	2.3732	2.8139	3.3336	3.9461	4.6673	5.5160	7.6861	10.6766
36	2.0399	2.4325	2.8983	3.4503	4.1039	4.8774	5.7918	8.1473	11.4239
37	2.0807	2.4933	2.9852	3.5710	4.2681	5.0969	6.0814	8.6361	12.2236
38	2.1223	2.5557	3.0748	3.6960	4.4388	5.3262	6.3855	9.1543	13.0793
39	2.1647	2.6196	3.1670	3.8254	4.6164	5.5659	6.7048	9.7035	13.9948
40	2.2080	2.6851	3.2620	3.9593	4.8010	5.8164	7.0400	10.2857	14.9745
41	2.2522	2.7522	3.3599	4.0978	4.9931	6.0781	7.3920	10.9029	16.0227
42	2.2972	2.8210	3.4607	4.2413	5.1928	6.3516	7.7616	11.5570	17.1443
43	2.3432	2.8915	3.5645	4.3897	5.4005	6.6374	8.1497	12.2505	18.3444
44	2.3901	2.9638	3.6715	4.5433	5.6165	6.9361	8.5572	12.9855	19.6285
45	2.4379	3.0379	3.7816	4.7024	5.8412	7.2482	8.9850	13.7646	21.0025
46	2.4866	3.1139	3.8950	4.8669	6.0748	7.5744	9.4343	14.5905	22.4726
47	2 5363	3.1917	4.0119	5.0373	6.3178	7.9153	9.9060	15.4659	24.0457
48	2.5871	3.2715	4.1323	5.2136	6.5705	8.2715	10.4013	16.3939	25.7289
49	2.6388	3.3533	4.2562	5.3961	6.8333	8.6437	10.9213	17.3775	27.5299
50	2.6916	3.4371	4.3839	5.5849	7.1067	9.0326	11.4674	18.4202	29.4570

PRESENT VALUE OF ONE DOLLAR DUE AT THE END OF n YEARS

n	2 %	2½%	3 %	3½%	4 %	4½%	5 %	6 %	7 %
1	.98039	.97561	.97087	.96618	.96154	.95694	.95238	.94340	.93458
2	.96117	.95181	.94260	.93351	.92456	.91573	.90703	.89000	.87344
3	.94232	.92860	.91514	.90194	.88900	.87630	.86384	.83962	.81630
4	.92385	.90595	.88849	.87144	.85480	.83856	.82270	.79209	.76290
5	.90573	.88385	.86261	.84197	.82193	.80245	.78353	.74726	.71299
6	.88797	.86230	.83748	.81350	.79031	.76790	.74622	.70496	.66634
7	.87056	.84127	.81309	.78599	.75992	.73483	.71068	.66506	.62275
8	.85349	.82075	.78941	.75941	.73069	.70319	.67684	.62741	.58201
9	.83676	.80073	.76642	.73373	.70259	.67290	.64461	.59190	.54393
10	.82035	.78120	.74409	.70892	.67556	.64393	.61391	.55839	.50835
11	.80426	.76214	.72242	.68495	.64958	.61620	.58468	.52679	.47509
12	.78849	.74356	.70138	.66178	.62460	.58966	.55684	.49697	.44401
13	.77303	.72542	.68095	.63940	.60057	.56427	.53032	.46884	.41496
14	.75788	.70773	.66112	.61778	.57748	.53997	.50507	.44230	.38782
15	.74301	.69047	.64186	.59689	.55526	.51672	.48102	.41727	.36245
16	.72845	.67362	.62317	.57671	.53391	.49447	.45811	.39365	.33873
17	.71416	.65720	.60502	.55720	.51337	.47318	.43630	.37136	.31657
18	.70016	.64117	.58739	.53836	.49363	.45280	.41552	.35034	.29586
19	.68643	.62553	.57029	.52016	.47464	.43330	.39573	.33051	.27651
20	.67297	.61027	.55368	.50257	.45639	.41464	.37689	.31180	.25842
21	.65978	.59539	.53755	.48557	.43883	.39679	.35894	.29416	.24151
22	.64684	.58086	.52189	.46915	.42196	.37970	.34185	.27751	.22571
23	.63416	.56670	.50669	.45329	.40573	.36335	.32557	.26180	.21095
24	.62172	.55288	.49193	.43796	.39012	.34770	.31007	.24698	.19715
25	.60953	.53939	.47761	.42315	.37512	.33273	.29530	.23300	.18425
26	.59758	.52623	.46369	.40884	.36069	.31840	.28124	.21981	.17220
27	.58586	.51340	.45019	.39501	.34682	.30469	.26785	.20737	.16093
28	.57437	.50088	.43708	.38165	.33348	.29157	.25509	.19563	.15040
29	.56311	.48866	.42435	.36875	.32065	.27902	.24295	.18456	.14056
30	.55207	.47674	.41199	.35628	.30832	.26700	.23138	.17411	.13137
31	.54125	.46511	.39999	.34423	.29646	.25550	.22036	.16425	.12277
32	.53063	.45377	.38834	.33259	.28506	.24450	.20987	.15496	.11474
33	.52023	.44270	.37703	.32134	.27409	.23397	.19987	.14619	.10723
34	.51003	.43191	.36604	.31048	.26355	.22390	.19035	.13791	.10022
35	.50003	.42137	.35538	.29998	.25342	.21425	.18129	.13011	.09366
36	.49022	.41109	.34503	.28983	.24367	.20503	.17266	.12274	.08754
37	.48061	.40107	.33498	.28003	.23430	.19620	.16444	.11580	.08181
38	.47119	.39128	.32523	.27056	.22529	.18775	.15661	.10924	.07646
39	.46195	.38174	.31575	.26141	.21662	.17967	.14915	.10306	.07146
40	.45289	.37243	.30656	.25257	.20829	.17193	.14205	.09722	.06678
41	.44401	.36335	.29763	.24403	.20028	.16453	.13528	.09172	.06241
42	.43530	.35448	.28896	.23578	.19257	.15744	.12884	.08653	.05833
43	.42677	.34584	.28054	.22781	.18517	.15066	.12270	.08163	.05451
44	.41840	.33740	.27237	.22010	.17805	.14417	.11686	.07701	.05095
45	.41020	.32917	.26444	.21266	.17120	.13796	.11130	.07265	.04761
46	.40215	.32115	.25674	.20547	.16461	.13202	.10600	.06854	.04450
47	.39427	.31331	.24926	.19852	.15828	.12634	.10095	.06466	.04159
48	.38654	.30567	.24200	.19181	.15219	.12090	.09614	.06100	.03887
49	.37896	.29822	.23495	.18532	.14634	.11569	.09156	.05755	.03632
50	.37153	.29094	.22811	.17905	.14071	.11071	.08720	.05429	.03395

Table IIIc—Amount of an Annuity

AMOUNT OF AN ANNUITY OF ONE DOLLAR PER YEAR AFTER n YEARS

n	2 %	2½ %	3 %	3½ %	4 %	4½ %	5 %	6 %	7 %
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0200	2.0250	2.0300	2.0350	2.0400	2.0450	2.0500	2.0600	2.0700
3	3.0604	3.0756	3.0909	3.1062	3.1216	3.1370	3.1525	3.1836	3.2149
4	4.1216	4.1525	4.1836	4.2149	4.2465	4.2782	4.3101	4.3746	4.4399
5	5.2040	5.2563	5.3091	5.3625	5.4163	5.4707	5.5256	5.6371	5.7507
6	6.3081	6.3877	6.4684	6.5502	6.6330	6.7169	6.8019	6.9753	7.1533
7	7.4343	7.5474	7.6625	7.7794	7.8983	8.0192	8.1420	8.3938	8.6540
8	8.5830	8.7361	8.8923	9.0517	9.2142	9.3800	9.5491	9.8975	10.2598
9	9.7546	9.9545	10.1591	10.3685	10.5828	10.8021	11.0266	11.4913	11.9780
10	10.9497	11.2034	11.4639	11.7314	12.0061	12.2882	12.5779	13.1808	13 8164
11	12.1687	12.4835	12.8078	13.1420	13.4864	13.8412	14.2068	14.9716	15.7836
12	13.4121	13.7956	14.1920	14.6020	15.0258	15.4640	15.9171	16.8699	17.8885
13	14.6803	15.1404	15.6178	16.1130	16.6268	17.1599	17.7130	18.8821	20.1406
14	15.9739	16.5190	17.0863	17.6770	18.2919	18.9321	19.5986	21.0151	22.5505
15	17.2934	17.9319	18 5989	19.2957	20.0236	20.7841	21.5786	23.2760	25.1290
16	18.6393	19.3802	20.1569	20.9710	21.8245	22.7193	23.6575	25.6725	27.8881
17	20.0121	20.8647	21.7616	22.7050	23.6975	24.7417	25.8404	28.2129	30.8402
18	21.4123	22.3863	23.4144	24.4997	25.6454	26.8551	28.1324	30.9057	33 9990
19	22.8406	23.9460	25.1169	26.3572	27.6712	29.0636	30.5390	33.7600	37 3790
20	24.2974	25.5447	26.8704	28.2797	29.7781	31.3714	33.0660	36.7856	40.9955
21	25.7833	27.1833	28.6765	30.2695	31.9692	33.7831	35.7193	39.9927	44.8652
22	27.2990	28.8629	30.5368	32.3289	34.2480	36.3034	38.5052	43.3923	49.0057
23	28.8450	30.5844	32.4529	34.4604	36.6179	38.9370	41.4305	46.9958	53.4361
24	30.4219	32.3490	34.4265	36.6665	39.0826	41.6892	44.5020	50.8156	58.1767
25	32.0303	34.1578	36.4593	38.9499	41.6459	44.5652	47.7271	54.8645	63.2490
26	33 6709	36.0117	38.5530	41.3131	44.3117	47.5706	51.1135	59.1561	68.6765
27	35.3443	37.9120	40.7096	43.7591	47.0842	50.7113	54.6691	63.7058	74.4838
28	37.0512	39.8598	42.9309	46.2906	49.9676	53.9933	58.4026	68.5281	80.6977
29	38.7922	41.8563	45.2189	48.9108	52.9663	57.4230	62.3227	73.6398	87.3465
30	40.5681	43.9027	47.5754	51.6227	56.0849	61.0071	66.4388	79.0582	94.4608
31	42.3794	46.0003	50.0027	54.4295	59.3283	64.7524	70.7608	84.8017	102.0730
32	44.2270	48.1503	52.5028	57.3345	62.7015	68.6662	75.2988	90.8898	110.2182
33	46.1116	50.3540	55.0778	60.3412	66.2095	72.7562	80.0638	97.3432	118.9334
34	48.0338	52.6129	57.7302	63.4532	69.8579	77.0303	85.0670	104.1838	128.2588
35	49.9945	54.9282	60.4621	66.6740	73.6522	81.4966	90.3203	111.4348	138.2369
36	51.9944	57.3014	63.2759	70.0076	77.5983	86.1640	95.8363	119.1209	148.9135
37	54.0343	59.7339	66.1742	73.4579	81.7022	91.0413	101.6281	127.2681	160.3374
38	56.1149	62.2273	69.1594	77.0289	85.9703	96.1382	107.7095	135.9042	172.5610
39	58.2372	64.7830	72.2342	80.7249	90.4091	101.4644	114.0950	145.0585	185.6403
40	60.4020	67.4026	75.4013	84.5503	95.0255	107.0303	120.7998	154.7620	199.6351
41	62.6100	70.0876	78.6633	88.5095	99.8265	112.8467	127.8398	165.0477	214.6096
42	64.8622	72.8398	82.0232	92.6074	104.8196	118.9248	135.2318	175.9505	230.6322
43	67.1595	75.6608	85.4839	96.8486	110.0124	125.2764	142.9933	187.5076	247.7765
44	69.5027	78.5523	89.0484	101.2383	115.4129	131.9138	151.1430	199.7580	266.1209
45	71.8927	81.5161	92.7199	105.7817	121.0294	138.8500	159.7002	212.7435	285.7493
46	74.3306	84.5540	96.5015	110.4840	126.8706	146.0982	168.6852	226.5081	306.7518
47	76.8172	87.6679	100.3965	115.3510	132.9454	153.6726	178.1194	241.0986	329.2244
48	79.3535	90.8596	104.4084	120.3883	139.2632	161.5879	188.0254	256.5645	353.2701
49	81.9406	94.1311	108.5406	125.6018	145.8337	169.8594	198.4267	272.9584	378.9990
50	84.5794	97.4843	112.7969	130.9979	152.6671	178.5030	209.3480	290.3359	406.5289

PRESENT VALUE OF ONE DOLLAR PER YEAR FOR n YEARS

n	2 %	2½ %	3 %	3½ %	4 %	4½ %	5 %	6 %	7 %
1	.9804	.9756	.9709	.9662	.9615	.9569	.9524	.9434	.9346
2	1.9416	1.9274	1.9135	1.8997	1.8861	1.8727	1.8594	1.8334	1.8080
3	2.8839	2.8560	2.8286	2.8016	2.7751	2.7490	2.7232	2.6730	2.6243
4	3.8077	3.7620	3.7171	3.6731	3.6299	3.5875	3 5460	3.4651	3.3872
5	4.7135	4.6458	4.5797	4.5151	4.4518	4.3900	4.3295	4.2124	4.1002
6	5.6014	5.5081	5.4172	5.3286	5.2421	5.1579	5.0757	4.9173	4.7665
7	6.4720	6.3494	6.2303	6.1145	6.0021	5.8927	5.7864	5.5824	5.3893
8	7.3255	7.1701	7.0197	6.8740	6.7327	6.5959	6.4632	6.2098	5.9713
9	8.1622	7.9709	7.7861	7.6077	7.4353	7.2688	7.1078	6 8017	6.5152
10	8.9826	8.7521	8.5302	8.3166	8.1109	7.9127	7.7217	7.3601	7.0236
11	9.7868	9.5142	9.2526	9.0016	8.7605	8.5289	8.3064	7.8869	7.4987
12	10.5753	10.2578	9.9540	9.6633	9.3851	9.1186	8.8633	8.3838	7.9427
13	11.3484	10.9832	10.6350	10.3027	9.9856	9.6829	9.3936	8.8527	8.3577
14	12.1062	11.6909	11.2961	10.9205	10.5631	10.2228	9.8986	9.2950	8.7455
15	12.8493	12.3814	11.9379	11.5174	11.1184	10.7395	10.3797	9.7122	9.1079
16	13.5777	13.0550	12.5611	12.0941	11.6523	11.2340	10.8378	10.1059	9.4466
17	14.2919	13.7122	13.1661	12.6513	12.1657	11.7072	11.2741	10.4773	9.7632
18	14.9920	14.3534	13.7535	13.1897	12.6593	12.1600	11.6896	10.8276	10.0591
19	15.6785	14.9789	14.3238	13.7098	13.1339	12.5933	12.0853	11.1581	10.3356
20	16.3514	15.5892	14.8775	14.2124	13.5903	13.0079	12.4622	11.4699	10.5940
21	17.0112	16.1845	15.4150	14.6980	14.0292	13.4047	12.8212	11.7641	10.8355
22	17.6580	16.7654	15.9369	15.1671	14.4511	13.7844	13.1630	12.0416	11.0612
23	18.2922	17.3321	16.4436	15.6204	14.8568	14.1478	13.4886	12.3034	11.2722
24	18.9139	17.8850	16.9355	16.0584	15.2470	14.4955	13.7986	12.5504	11.4693
25	19.5235	18.4244	17.4131	16.4815	15.6221	14.8282	14.0939	12.7834	11.6536
26	20.1210	18.9506	17.8768	16.8904	15.9828	15.1466	14.3752	13.0032	11.8258
27	20.7069	19.4640	18.3270	17.2854	16.3296	15.4513	14.6430	13.2105	11.9867
28	21.2813	19.9649	18.7641	17.6670	16.6631	15.7429	14.8981	13.4062	12.1371
29	21.8444	20.4535	19.1885	18.0358	16.9837	16.0219	15.1411	13.5907	12 2777
30	22.3965	20.9303	19.6004	18.3920	17.2920	16.2889	15.3725	13.7648	12.4090
31	22.9377	21.3954	20.0004	18.7363	17.5885	16.5444	15.5928	13.9291	12.5318
32	23.4683	21.8492	20.3888	19.0689	17.8736	16.7889	15.8027	14.0840	12.6466
33	23.9886	22.2919	20.7658	19.3902	18.1476	17.0229	16.0025	14.2302	12.7538
34	24.4986	22.7238	21.1318	19.7007	18.4112	17.2468	16.1929	14.3681	12.8540
35	24.9986	23.1452	21.4872	20 0007	18.6646	17.4610	16.3742	14.4982	12.9477
36	25.4888	23.5563	21.8323	20.2905	18.9083	17.6660	16.5469	14.6210	13.0352
37	25.9695	23.9573	22.1672	20.5705	19 1426	17.8622	16.7113	14.7368	13.1170
38	26.4406	24.3486	22.4925	20.8411	19.3679	18.0500	16.8679	14.8460	13.1935
39	26.9026	24.7303	22.8082	21.1025	19.5845	18.2297	17.0170	14.9491	13.2649
40	27.3555	25.1028	23.1148	21.3551	19.7928	18.4016	17.1591	15.0463	13.3317
41	27.7995	25.4661	23.4124	21.5991	19.9931	18.5661	17.2944	15.1380	13.3941
42	28.2348	25.8206	23.7014	21.8349	20.1856	18.7236	17.4232	15.2245	13.4524
43	28.6616	26.1664	23.9819	22.0627	20.3708	18.8742	17.5459	15.3062	13.5070
44	29.0800	26.5038	24.2543	22.2828	20.5488	19.0184	17.6628	15.3832	13.5579
45	29.4902	26.8330	24.5187	22.4955	20.7200	19.1563	17.7741	15.4558	13.6055
46	29.8923	27.1542	24.7754	22.7009	20.8847	19.2884	17.8801	15.5244	13.6500
47	30.2866	27.4675	25.0247	22.8994	21.0429	19.4147	17.9810	15.5890	13.6910
48	30.6731	27.7732	25.2667	23.0912	21.1951	19.5356	18.0772	15.6500	13.7305
49	31.0521	28.0714	25.5017	23.2766	21.3415	19.6513	18.1687	15.7076	13.7668
50	31.4236	28.3623	25.7298	23.4556	21.4822	19.7620	18.2559	15.7619	13.8007

r	$1+r$	$\log(1+r)$
¼ %	1.005	00216 60617 36508
½ %	1.010	00432 13737 82643
¾ %	1.015	00646 60422 49282
2 %	1.020	00860 01717 61918
2½ %	1.025	01072 38633 91773
3 %	1.030	01283 72247 05172
3½ %	1.035	01494 03497 92937
4 %	1.040	01703 33392 98780
4½ %	1.045	01911 62904 47073
5 %	1.050	02118 92990 03838

r	$1+r$	$\log(1+r)$
5½ %	1.055	02325 24596 33711
6 %	1.060	02530 58652 64770
6½ %	1.065	02734 98077 74757
7 %	1.070	02938 37776 85210
7½ %	1.075	03140 84642 51624
8 %	1.080	03342 37554 80950
8½ %	1.085	03542 97381 84548
9 %	1.090	03742 64979 40624
9½ %	1.095	03941 41191 76137
10 %	1.100	04139 26851 58225

For Amount, A, of any principal, P, after n years: $A = P(1+r)^n$

For present worth, P, of any amount, A, at the end of n years: $P = A \div (1+r)^n$

Table IIIf—American Experience Mortality Table

Based on 100,000 living at age 10

At Age	Number Surviving	Deaths	At Age	Number Surviving	Deaths	At Age	Number Surviving	Deaths	At Age	Number Surviving	Deaths
10	100,000	749	35	81,822	762	60	57,917	1,546	85	5,485	1,292
11	99,251	746	36	81,060	767	61	56,371	1,628	86	4,193	1,114
12	98,505	745	37	80,293	772	62	54,743	1,715	87	3,079	953
13	97,762	749	38	79,611	749	63	53,030	1,800	88	2,146	744
14	97,022	757	39	78,862	756	64	51,230	1,889	89	1,402	555
15	96,285	765	40	78,106	765	65	49,341	1,980	90	847	385
16	95,550	772	41	77,341	774	66	47,361	2,070	91	462	246
17	94,818	779	42	76,567	785	67	45,291	2,158	92	216	137
18	94,089	787	43	75,782	797	68	43,133	2,245	93	79	58
19	93,362	795	44	74,985	812	69	40,890	2,321	94	21	18
20	92,637	723	45	74,173	828	70	38,569	2,391	95	3	3
21	91,914	722	46	73,345	848	71	36,178	2,448			
22	91,192	721	47	72,497	870	72	33,730	2,487			
23	90,471	720	48	71,627	896	73	31,243	2,505			
24	89,751	719	49	70,731	927	74	28,738	2,501			
25	89,032	718	50	69,804	962	75	26,237	2,476			
26	88,314	718	51	68,842	1,001	76	23,761	2,431			
27	87,596	718	52	67,841	1,044	77	21,330	2,369			
28	86,878	718	53	66,797	1,091	78	18,961	2,291			
29	86,160	719	54	65,706	1,143	79	16,670	2,196			
30	85,441	720	55	64,563	1,199	80	14,474	2,091			
31	84,721	721	56	63,364	1,260	81	12,383	1,964			
32	84,000	723	57	62,104	1,325	82	10,419	1,816			
33	83,277	726	58	60,779	1,394	83	8,603	1,648			
34	82,551	729	59	59,385	1,468	84	6,955	1,470			

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